Borel–Weil Theory for Root Graded Banach–Lie Groups

Christoph Müller[*], Karl-Hermann Neeb[†], Henrik Seppänen[‡]

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Abstract

In this paper we introduce (weakly) root graded Banach–Lie algebras and corresponding Lie groups as natural generalizations of group like $\mathrm{GL}_n(A)$ for a Banach algebra A or groups like $C(X, K)$ of continuous maps of a compact space X into a complex semisimple Lie group K. We study holomorphic induction from holomorphic Banach representations of so-called parabolic subgroups P to representations of G on holomorphic sections of homogeneous vector bundles over G/P. One of our main results is an algebraic characterization of the space of sections which is used to show that this space actually carries a natural Banach structure, a result generalizing the finite dimensionality of spaces of sections of holomorphic bundles over compact complex manifolds. We also give a geometric realization of any irreducible holomorphic representation of a (weakly) root graded Banach–Lie group G and show that all holomorphic functions on the spaces G/P are constant.

Keywords: Banach–Lie group, holomorphic vector bundle, induced representation

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Introduction

Let $\mathfrak{g}_\Delta$ be the finite-dimensional semisimple complex Lie algebra with root system Δ and fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_\Delta$. A complex Banach–Lie algebra $\mathfrak{g}$ is said to be weakly Δ-graded if it contains $\mathfrak{g}_\Delta$ and decomposes as a direct sum

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha$$

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of simultaneous ad $\mathfrak{h}$-eigenspaces. It is called *root graded* if, in addition, $\Delta = \mathcal{R}$ and $\mathfrak{g}$ is topologically generated by the root spaces $\mathfrak{g}_\alpha$.

The systematic study of root graded Lie algebras (with irreducible Δ) was initiated by Berman and Moody in [BM92], where they studied Lie algebras graded by root systems of the types A, D and E. Corresponding results for non-simply laced root systems have been obtained by Benkart and Zelmanov in [BZ96]. The classification of root graded Lie algebras in the algebraic context was completed by Allison, Benkart and Gao in [ABG00]. The main point of the classification is to associate to a Δ-graded Lie algebra $\mathfrak{g}$ a coordinate algebra $\mathcal{A}$ of a certain type depending on Δ, and then to show that, up to central extensions, $\mathfrak{g}$ is determined by its coordinate algebra. For type A_1, the coordinate algebras turn out to be unital Jordan algebras, for A_2 unital alternative algebras, and for A_n, $n > 2$, they are unital associative algebras. For types D and E they are commutative, so that $\mathfrak{g}$ is a central extension of $\mathcal{A} \otimes \mathfrak{g}_\Delta$. Apart from simple complex Lie algebras, the most well-known class of root graded Lie algebras are affine Kac–Moody algebras ([Ka90, Ch. 6]). For the case $\mathcal{R} = BC_r$ and Δ of type B, C or D, we refer to the memoir [ABG02]. The classification scheme for root graded Lie algebras has been extended to the topological context of locally convex Lie algebras in [Ne03] to cover many classes arising mathematical physics, operator theory and geometry.

This paper is the first in a series dedicated to various aspects of holomorphic representations of weakly root graded Banach–Lie groups G, i.e., groups whose Lie algebra $\mathfrak{g}$ is weakly root graded. Since there is a natural notion of parabolic subgroups P of a root graded Lie group G, and the corresponding homogeneous spaces G/P carry complex manifold structures, it is a natural problem to understand the representations of G in spaces of holomorphic sections of homogeneous holomorphic vector bundles $\mathbb{E}_\rho = G \times_\rho E$ over G/P, defined by a holomorphic representation $\rho \colon P \to \mathrm{GL}(E)$, where E is a Banach space and ρ is a morphism of Banach–Lie groups.

The classical context for these problems is the Borel–Weil Theorem, where G is a complex reductive Lie group and G/P is a generalized flag manifold, hence in particular compact. In this case the space of holomorphic sections is always finite dimensional if E is so. The Borel–Weil Theorem identifies those holomorphic characters $\rho \colon P \to \mathbb{C}^\times = \mathrm{GL}(\mathbb{C})$ for which this space is non-zero, and the natural G representation on this space, which is irreducible.

One of the main results of this paper is that for each holomorphic Banach representation (ρ, E) of a parabolic subgroup P, the space of holomorphic sections of $\mathbb{E}_\rho$, which we identify with the space

$$\mathcal{O}_\rho(G, E) = \{f \in \mathcal{O}(G, E) \colon (\forall g \in G)(\forall p \in P)\ f(gp) = \rho(p)^{-1} f(g)\}$$

carries a natural Banach space structure for which the representation of G, defined by $(\pi(g)f)(x) = f(g^{-1}x)$ defines a morphism $\pi \colon G \to \mathrm{GL}(\mathcal{O}_\rho(G, E))$ of Banach–Lie groups (Theorem 3.7). Here the remarkable point is that on the much larger space $\mathcal{O}(G, E)$ of all E-valued holomorphic functions, there is no natural locally convex structure for which the G-action is continuous, resp., holomorphic. A natural topology on this space is the compact open topology, i.e., the

topology of uniform convergence on compact subsets of G. With respect to this topology, $\mathcal{O}(G, E)$ is a complete locally convex space ([Ne01, Thm. III.11(c)]), but for infinite dimensional groups the action of G is not continuous. However, the finite dimensional complex subgroup G_Δ corresponding to the subalgebra $\mathfrak{g}_\Delta$ of $\mathfrak{g}$ acts holomorphically on this space. An instructive example illustrating the problem is the action of a Banach space X, considered as an abelian Banach–Lie group, on the space A of affine holomorphic functions $X \to \mathbb{C}$ by $\pi(x)(f)(y) = f(y - x)$. This action is continuous with respect to the compact open topology if and only if the evaluation map $X \times X' \to \mathbb{C}, (x, f) \mapsto f(x)$ is continuous, but this is equivalent to X being finite dimensional because otherwise the bipolars of compact subsets of X are never 0-neighborhoods (cf. [KM97, p.2], [Mue06, 2.2.10-13]). Our strategy to find a better topology on the space $\mathcal{O}_\rho(G, E)$ is to realize it, basically by Taylor expansion in $\mathbf{1}$, as a set of linear maps $\alpha \colon U(\mathfrak{g}) \to E$ on the enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G. These linear maps are always *continuous* in the sense that all n-linear maps $\alpha_n \colon \mathfrak{g}^n \to E, (x_1, \ldots, x_n) \mapsto \alpha(x_1 \cdots x_n)$ are continuous, and the space $\mathrm{Hom}(U(\mathfrak{g}), E)_c$ of all continuous linear maps carries a natural Fréchet space structure with respect to which the natural $\mathfrak{g}$ action is a continuous bilinear map (Section 2). The key observation is that in the coinduced representation $\mathrm{Hom}_\mathfrak{p}(U(\mathfrak{g}), E)_c$ corresponding to the $\mathfrak{p}$-module E, the space of $\mathfrak{g}_\Delta$-finite vectors decomposes into finitely many $\mathfrak{h}$-weight spaces. Based on this observation, we proceed to show that it is closed in the aforementioned Fréchet space and that it even is a Banach space. This provides the infinitesimal picture. The bridge to the G-action on $\mathcal{O}_\rho(G, E)$ is developed in Section 3. It is based on an application of the general Peter–Weyl Theorem to the action of a compact real form of the semisimple complex Lie group G_Δ on the locally convex spaces $\mathcal{O}_\rho(G, E)$, endowed with the compact open topology. If G is 1-connected and P is connected, we show that the map

$$\Phi \colon \mathcal{O}_\rho(G, E) \to \mathrm{Hom}_\mathfrak{p}(U(\mathfrak{g}), E)_c, \quad \Phi(f)(D) := (D_r f)(\mathbf{1}),$$

where D_r is the right invariant differential operator on G associated to $D \in U(\mathfrak{g})$, defines a bijection onto the Banach subspace of $\mathfrak{g}_\Delta$-finite elements in the Fréchet space $\mathrm{Hom}_\mathfrak{p}(U(\mathfrak{g}), E)_c$ (Theorem 3.7). This result has some immediate consequences that are derived in Section 3. One is that all holomorphic functions on G/P are constant, although this manifold is far from being compact. Another one is that $\mathcal{O}_\rho(G, E)$ is finite dimensional if G and E are. The proof is rather elementary and does not rely on any deep theory of elliptic operators or sheaves.

In connected complex reductive Lie groups, the parabolic subgroups are always connected. This is no longer the case for root graded Lie groups, so that one has to understand the influence of the passage from a holomorphic representation $\rho \colon P \to \mathrm{GL}(E)$ to the restriction $\rho_0 := \rho|_{P_0}$ to its identity component. Clearly, $\mathcal{O}_\rho(P, E) \subseteq \mathcal{O}_{\rho_0}(P, E)$, but it is also natural to start with a representation ρ_0 of P_0 and ask for extensions to P for which $\mathcal{O}_\rho(G, P)$ is non-zero. In Theorem 4.6 we give a complete answer to this question in the most important case, where ρ_0 is irreducible with $\mathrm{End}_{P_0}(E) = \mathbb{C}\mathbf{1}$. We show that if all representations $p.\rho_0$, defined by $(p.\rho_0)(x) := \rho_0(p^{-1}xp)$ are equivalent to ρ_0 (which

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is necessary for the existence of some extension ρ) and $\mathcal{O}_{\rho_0}(G, E)$ is non-zero, then there exists a unique extension ρ to P with $\mathcal{O}_\rho(P, E) = \mathcal{O}_{\rho_0}(G, P)$, and for all other extensions γ of ρ_0, the space $\mathcal{O}_\gamma(G, P)$ vanishes. We find this quite surprising. It reduces all questions on the representations $\mathcal{O}_\rho(G, E)$ to the case where P is connected and G is simply connected, and then Φ maps it isomorphically to $\mathrm{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)_c^{[\mathfrak{g}_\Delta]}$. In Section 5 we show that for each connected parabolic subgroup P, each irreducible holomorphic representation of G can be realized in some space $\mathcal{O}_\rho(G, E)$ for an irreducible holomorphic representation (ρ, E) of P. The difficult question that remains open at this point is a characterization of those irreducible holomorphic representations (ρ, E) of P for which $\mathcal{O}_\rho(G, E)$ is non-zero, which implies the existence of a corresponding irreducible holomorphic G-representation sitting as a minimal non-zero submodule in $\mathcal{O}_\rho(G, E)$.

In an appendix we develop a quite general variant of Frobenius Reciprocity for representations of G on locally convex spaces for which all orbit maps are holomorphic. This applies in particular to the representation on $\mathcal{O}(G, E)$, endowed with the topology of pointwise convergence.

We plan to continue this project in subsequent papers which address the special cases where G is finite dimensional but not necessarily semisimple, such as $\mathrm{GL}_n(A)$ for a finite dimensional algebra A, and where $\dim E = 1$ and $G = \mathrm{GL}_n(A)$ for a Banach algebra A, resp., $\mathfrak{g} = A \otimes \mathfrak{g}_\Delta$ for a commutative Banach algebra A. The main result is a characterization of those homogeneous line bundles which admit global holomorphic sections. It turns out that these are generated by pullbacks of positive line bundles over compact flag manifolds G/P with respect to natural mappings induced by characters of the algebra A.

Several results presented in the present paper grew out of predecessors from Ch. Müller's thesis [Mue06] which deals only with the scalar case $E = \mathbb{C}$.

1 Root graded Banach–Lie groups

In this section we introduce weakly root graded Lie algebras and their parabolic subalgebras. We also discuss parabolic subgroups of corresponding Banach–Lie groups and how they can be used to obtain triple coordinates of large open identity neighborhoods.

Root graded Lie algebras and parabolic subalgebras

Definition 1.1 Let Δ be a finite reduced root system and $\mathfrak{g}_\Delta$ be the corresponding finite dimensional complex semisimple Lie algebra. A complex Banach–Lie algebra $\mathfrak{g}$ is said to be *weakly Δ-graded* if the following conditions are satisfied:

(R1) $\mathfrak{g}_\Delta$ is a Lie subalgebra of $\mathfrak{g}$.

(R2) For some (and hence for each) Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_\Delta$, $\mathfrak{g}$ decomposes as a finite direct sum of $\mathfrak{h}$-eigenspaces $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha =$

$$\{x \in \mathfrak{g} \colon (\forall h \in \mathfrak{h})\, [h,x] = \alpha(h)x\}.$$

The subalgebra $\mathfrak{g}_\Delta$ of $\mathfrak{g}$ is called a *grading subalgebra*. We say that $\mathfrak{g}$ is *(weakly) root graded* if $\mathfrak{g}$ is (weakly) Δ-graded for some Δ. If, in addition, $\mathcal{R} = \Delta$ and $\sum_{\alpha \in \Delta}[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is dense in $\mathfrak{g}_0$, then $\mathfrak{g}$ is said to be Δ-*graded*.

For $\alpha \in \Delta$, the unique element $\check{\alpha} \in [\mathfrak{g}_{\Delta,\alpha}, \mathfrak{g}_{\Delta,-\alpha}]$, satisfying $\alpha(\check{\alpha}) = 2$ is called the *coroot* corresponding to α. From the representation theory of $\mathfrak{sl}_2(\mathbb{C})$, it follows that $\mathcal{R}(\check{\alpha}) \subseteq \mathbb{Z}$ for each coroot, and it is well known that $\mathfrak{h} = \operatorname{span}\check{\Delta}$.

Examples 1.2 (a) Let Δ be a reduced finite root system and $\mathfrak{g}_\Delta$ be the corresponding semisimple complex Lie algebra. If A is a commutative unital Banach algebra, then $\mathfrak{g} := A \otimes \mathfrak{g}_\Delta$ is a Δ-graded Banach–Lie algebra with respect to the bracket

$$[a \otimes x, a' \otimes x'] := aa' \otimes [x,x'].$$

The embedding $\mathfrak{g}_\Delta \hookrightarrow \mathfrak{g}$ is given by $x \mapsto \mathbf{1} \otimes x$.

(b) If A is a unital Banach algebra, then the $(n \times n)$-matrix algebra $M_n(A) \cong A \otimes M_n(\mathbb{C})$ also is a Banach algebra. We write $\mathfrak{gl}_n(A)$ for this algebra, endowed with the commutator bracket. Then $\mathfrak{gl}_n(A)$ is a weakly A_{n-1}-graded Lie algebra with grading subalgebra $\mathfrak{g}_\Delta = \mathbf{1} \otimes \mathfrak{sl}_n(\mathbb{C})$.

(c) Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ be a 3-graded Banach–Lie algebra for which there exist elements $e \in \mathfrak{g}_1$ and $f \in \mathfrak{g}_{-1}$ such that $h := [e,f]$ satisfies

$$\mathfrak{g}_{\pm 1} = \{x \in \mathfrak{g} \colon [h,x] = \pm 2x\}.$$

Then $\mathfrak{g}$ is weakly A_1-graded with grading subalgebra $\mathfrak{g}_\Delta = \operatorname{span}\{e,f,h\}$ (cf. [BN04]).

If, more generally, $\mathfrak{g} = \sum_{j=-n}^{n} \mathfrak{g}_j$ is $(2n+1)$-graded and there exist $e \in \mathfrak{g}_2$ and $f \in \mathfrak{g}_{-2}$ such that $h := [e,f]$ satisfies

$$\mathfrak{g}_j = \{x \in \mathfrak{g} \colon [h,x] = jx\},$$

then $\mathfrak{g}$ is weakly A_1-graded.

(d) Assume that $\mathfrak{g}$ is weakly Δ-graded, and that V is a $\mathfrak{g}$-module which decomposes into a direct sum of finitely many weight spaces under $\mathfrak{h}$. Then the trivial abelian extension $V \rtimes \mathfrak{g}$ of $\mathfrak{g}$ by V defined as

$$[(v_1, x_1), (v_2, x_2)] := (x_1.v_2 + x_2.v_1, [x_1, x_2]), \quad v_1, v_2 \in V, x_1, x_2 \in \mathfrak{g}$$

is also weakly Δ-graded.

Remark 1.3 (Central extensions) (a) If $\mathfrak{g}$ is (weakly) Δ-graded, then its center $\mathfrak{z}(\mathfrak{g})$ is contained in $\mathfrak{g}_0$ and intersects $\mathfrak{g}_\Delta$ trivially, so that the Banach–Lie algebra $\mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \cong \operatorname{ad}\mathfrak{g}$ is also (weakly) Δ-graded.

(b) If $q \colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ is a central extension of the weakly Δ-graded Banach–Lie algebra $\mathfrak{g}$ with kernel $\mathfrak{z}$, then the adjoint action of $\mathfrak{g}$ on itself lifts to a natural action on $\widehat{\mathfrak{g}}$ for which q is equivariant. Since the central extension $q^{-1}(\mathfrak{g}_\Delta)$ of $\mathfrak{g}_\Delta$ splits, we may consider $\mathfrak{g}_\Delta$, and therefore also $\mathfrak{h}$, as a subalgebra of $\widehat{\mathfrak{g}}$.

Then the subspaces $q^{-1}(\mathfrak{g}_\alpha)$, $\alpha \in \mathcal{R}$, are $\mathfrak{h}$-invariant and contain $\mathfrak{z} \subseteq \widehat{\mathfrak{g}}_0$. If $x \in \mathfrak{h}$ satisfies $\alpha(x) = 1$, then $(\operatorname{ad} x - \mathbf{1})(q^{-1}(\mathfrak{g}_\alpha)) \subseteq \ker q = \mathfrak{z}$ implies that $\operatorname{ad} x(\operatorname{ad} x - \mathbf{1})$ vanishes on $q^{-1}(\mathfrak{g}_\alpha)$, hence that $\operatorname{ad} x$ is diagonalizable on this space with eigenvalues 0 and 1. Clearly $\mathfrak{z} \subseteq \ker \operatorname{ad} x$ and $[x, q^{-1}(\mathfrak{g}_\alpha)]$ maps surjectively onto $\mathfrak{g}_\alpha$, which leads to the direct decomposition

$$q^{-1}(\mathfrak{g}_\alpha) = \mathfrak{z} \oplus [x, q^{-1}(\mathfrak{g}_\alpha)].$$

Since $\mathfrak{h}$ is abelian, $[x, q^{-1}(\mathfrak{g}_\alpha)]$ is $\mathfrak{h}$-invariant, and we derive that

$$\widehat{\mathfrak{g}}_\alpha = \{y \in \widehat{\mathfrak{g}} \colon (\forall h \in \mathfrak{h})\, [h, y] = \alpha(h)y\} = [x, q^{-1}(\mathfrak{g}_\alpha)].$$

This proves that $\operatorname{ad} \mathfrak{h}$ is diagonalizable on the subspace $\mathfrak{z} + \sum_{\alpha \in \mathcal{R}} \widehat{\mathfrak{g}}_\alpha \subseteq \widehat{\mathfrak{g}}$. If we assume, in addition, that $\widehat{\mathfrak{g}}_0 := q^{-1}(\mathfrak{g}_0)$ is centralized by $\mathfrak{h}$ (which is automatic if $\widehat{\mathfrak{g}}$ is generated by the subspaces $\widehat{\mathfrak{g}}_\alpha$), then $\widehat{\mathfrak{g}}$ is also weakly Δ-graded.

For more details on other classes of examples and a discussion of topological issues related to root graded Lie algebras, we refer to [Ne03].

Definition 1.4 A subset $\Sigma \subseteq \mathcal{R}$ is called a *parabolic systems* if there exists an element $x \in \mathfrak{h}$ with
$$\Sigma = \{\alpha \in \mathcal{R} \colon \alpha(x) \geq 0\}. \tag{1}$$
For a parabolic system Σ, we put

$$\Sigma^+ := \Sigma \setminus -\Sigma, \quad \Sigma^0 := \Sigma \cap -\Sigma \quad \text{and} \quad \Sigma^- = \mathcal{R} \setminus \Sigma.$$

For each parabolic system Σ, $\mathfrak{p}_\Sigma := \mathfrak{g}_0 + \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha$ is a subalgebra of $\mathfrak{g}$, called the corresponding *parabolic subalgebra*.

A parabolic system Σ is called a *positive system* if $\Sigma^0 = \emptyset$. Since $\mathcal{R}$ is real-valued on $\mathfrak{h}_{\mathbb{R}} := \operatorname{span}_{\mathbb{R}} \check{\Delta}$, the set of all elements $y \in \mathfrak{h}_{\mathbb{R}}$ with $\mathcal{R}(y) \subseteq \mathbb{R}^\times$ is dense, and choosing y sufficiently close to the element x from above, it follows that each parabolic system Σ contains a positive systems $\mathcal{R}^+$.

Remark 1.5 Since $\mathcal{R}$ is finite, there exists for each parabolic system Σ an element $x_\Sigma \in \mathfrak{h}$ with $\Sigma^+(x_\Sigma) \geq 1$ and $\Sigma^-(x_\Sigma) \leq -1$.

Root graded Lie groups and parabolic subgroups

Definition 1.6 A Banach–Lie group G is said to be *(weakly) root graded* if its Lie algebra $\mathrm{L}(G)$ is (weakly) root graded.

Remark 1.7 (a) Not every Banach–Lie algebra $\mathfrak{g}$ is integrable in the sense that there exists a Banach–Lie group G with Lie algebra $\mathfrak{g}$. Since $\mathfrak{g} = C^1(\mathbb{S}^1, \mathfrak{sl}_2(\mathbb{C}))$ is A_1-graded with grading algebra $\mathfrak{sl}_2(\mathbb{C})$, and this property is inherited by the central extension defined by the $\mathfrak{sl}_2(\mathbb{C})$-invariant cocycle

$$\omega(\xi, \eta) = \int_0^{2\pi} \operatorname{tr}(\xi(t)\eta'(t))\, dt$$

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(cf. Remark 1.3), the discussion in [EK64] implies the existence of non-integrable root graded Banach–Lie algebras.

(b) If A is a commutative unital Banach algebra, then any inclusion $\mathfrak{g}_\Delta \subseteq \mathfrak{gl}_n(\mathbb{C})$ yields an inclusion $\mathfrak{g} \hookrightarrow A \otimes M_n(\mathbb{C}) \cong M_n(A)$ of Banach–Lie algebras, which implies the existence of a Lie group G with Lie algebra $\mathfrak{g}$ ([Ms62], [Ne06, Thm. IV.4.9]).

(c) For similar reasons, for each Banach–Lie algebra $\mathfrak{g}$, the quotient $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is integrable because it embeds continuously into the integrable Lie algebra $\operatorname{der}(\mathfrak{g}) = \mathrm{L}(\operatorname{Aut}(\mathfrak{g}))$ ([EK64], [Ne06]).

In the following, G is a weakly Δ-graded Banach–Lie group. If $\mathfrak{p}$ is a parabolic subalgebra, we write Σ for the corresponding parabolic system with $\mathfrak{p} = \mathfrak{g}_0 + \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha$. Then we have a semidirect decomposition

$$\mathfrak{p} = \mathfrak{u} \rtimes \mathfrak{l}, \quad \text{where} \quad \mathfrak{l} = \mathfrak{g}_0 + \sum_{\alpha \in \Sigma^0} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{u} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha.$$

That $\mathfrak{l}$ is a subalgebra and $\mathfrak{u}$ is an ideal in $\mathfrak{p}$ follows easily from the relations

$$(\Sigma^+ + \Sigma^+) \cap \mathcal{R} \subseteq \Sigma^+ \quad \text{and} \quad (\Sigma^0 + \Sigma^+) \cap \mathcal{R} \subseteq \Sigma^+, \tag{2}$$

which are obvious consequence of (1). The subalgebra $\mathfrak{u}$ is closed and nilpotent. In fact, if x_Σ is chosen as in Remark 1.5 and $N > \max \Sigma^+(x_\Sigma)$, then N-fold brackets in $\mathfrak{u}$ vanish. Likewise

$$\mathfrak{n} = \sum_{\alpha \in -\Sigma^+} \mathfrak{g}_\alpha$$

is a nilpotent closed subalgebra. As a Banach space, $\mathfrak{g}$ has the direct sum decompositions

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{l} \oplus \mathfrak{u} = \mathfrak{n} \oplus \mathfrak{p}. \tag{3}$$

Remark 1.8 Note that the subalgebra $\mathfrak{l}$ is weakly root graded with respect to the root system $\Delta^0 := \Sigma^0 \cap \Delta$ and the grading subalgebra

$$\mathfrak{g}_{\Delta^0} := \mathfrak{h}_{\Delta^0} + \sum_{\alpha \in \Delta^0} \mathfrak{g}_{\Delta,\alpha}, \quad \text{where} \quad \mathfrak{h}_{\Delta_0} := \operatorname{span}\{\breve{\alpha} \colon \alpha \in \Delta^0\}.$$

The intersection $\mathfrak{p}_\Delta := \mathfrak{p} \cap \mathfrak{g}_\Delta$ is a parabolic subalgebra of $\mathfrak{g}_\Delta$ satisfying

$$\mathfrak{p}_\Delta = \mathfrak{u}_\Delta \rtimes \mathfrak{l}_\Delta \quad \text{for} \quad \mathfrak{l}_\Delta := \mathfrak{l} \cap \mathfrak{g}_\Delta \quad \text{and} \quad \mathfrak{u}_\Delta := \mathfrak{u} \cap \mathfrak{g}_\Delta.$$

We also put $\mathfrak{n}_\Delta := \mathfrak{g}_\Delta \cap \mathfrak{n}$ and obtain the direct sum decomposition $\mathfrak{g}_\Delta = \mathfrak{n}_\Delta \oplus \mathfrak{p}_\Delta$.

If $\mathfrak{g}$ is a Lie algebra and $E \subseteq \mathfrak{g}$ a subspace, we write

$$\mathfrak{n}_\mathfrak{g}(E) := \{x \in \mathfrak{g} \colon [x, E] \subseteq E\}$$

for the *normalizer of E in $\mathfrak{g}$*.

Lemma 1.9 *For any parabolic subalgebra* $\mathfrak{p} = \mathfrak{u} \rtimes \mathfrak{l}$, *we have*

$$\mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}) = \mathfrak{p} \quad and \quad \mathfrak{n}_{\mathfrak{g}}(\mathfrak{l}) = \mathfrak{l}.$$

Proof.   (cf. [Mue06, Lemma 1.2.11]) Since $\mathfrak{p}$ and $\mathfrak{l}$ are $\mathfrak{h}$-invariant, their normalizers contains $\mathfrak{h}$, and hence are adapted to the root decomposition.

Let $\alpha \in \mathcal{R} \setminus \Sigma$ and $0 \neq y_\alpha \in \mathfrak{g}_\alpha$. From $y_\alpha \in [\mathfrak{h}, y_\alpha]$ it follows that y_α does not normalize $\mathfrak{p}$, which leads to $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}) = \mathfrak{p}$. We likewise derive that $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{l}) = \mathfrak{l}$.   ∎

Remark 1.10 In the case when $\mathcal{R} = \Delta$ we can also derive the equality $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{u}) = \mathfrak{p}$ from the representation theory of $\mathfrak{sl}_2(\mathbb{C})$. This does however not hold in general. Consider for example the semidirect product $\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathbb{C}^2$ given by the standard representation of $\mathfrak{sl}_2(\mathbb{C})$. Here $\mathcal{R} = \{\pm\alpha, \pm2\alpha\}$, and for the parabolic system $\Sigma = \{\alpha, 2\alpha\}$ we have $\mathfrak{u} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. However $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{u}) = \mathfrak{g}_{-\alpha} \oplus \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ since $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha] = 0$.

In the preceding lemma, we have seen that the closed Lie subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is self-normalizing. From Lemmas IV.11/12 in [Ne04] we thus obtain that

$$N_G(\mathfrak{p}) := \{g \in G \colon \operatorname{Ad}(g)\mathfrak{p} = \mathfrak{p}\}$$

is a Lie subgroup of G whose Lie algebra is

$$\mathbf{L}(N_G(\mathfrak{p})) = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}) = \mathfrak{p}.$$

Definition 1.11 In the following we call any open subgroup P of $N_G(\mathfrak{p})$ a *(standard) parabolic subgroup* of G. Then $P_0 = \langle\exp\mathfrak{p}\rangle$ is a connected Lie subgroup of G, and since $\mathfrak{n}$ is a closed complement of $\mathfrak{p}$, we see that P is a split Lie subgroup (cf. [Ne04, Def. IV.6]). Therefore the quotient space G/P carries a natural complex manifold structure for which the quotient map $q \colon G \to G/P, g \mapsto gP$ is a submersion with holomorphic local cross section.

Proposition 1.12 *For any parabolic subgroup P of G, we put*

$$U := \exp_G \mathfrak{u}, \quad N := \exp_G \mathfrak{n} \quad and \quad L := P \cap N_G(\mathfrak{l}) \cap N_G(\mathfrak{n}).$$

Then the following assertions hold:

(a) *U is a Lie subgroup with Lie algebra $\mathfrak{u}$ whose exponential function is a diffeomorphism.*

(b) *N is a Lie subgroup with Lie algebra $\mathfrak{n}$ whose exponential function is a diffeomorphism.*

(c) *L is a Lie subgroup with Lie algebra $\mathfrak{l}$ which acts holomorphically by conjugation on N and U.*

(d) *The multiplication map $U \rtimes L \to P, (u, l) \mapsto ul$ is biholomorphic onto an open subgroup of P.*

(e) *The multiplication map $N \times L \times U \to G$ is biholomorphic onto an open subset of G.*

Proof.   (a) Let $*$ denote the Baker–Campbell–Hausdorff multiplication on the nilpotent Lie algebra $\mathfrak{u}$, turning it into a simply connected complex Banach–Lie group $(\mathfrak{u}, *)$. Then the exponential function

$$\exp \colon \mathfrak{u} \to G, \quad x \mapsto \exp_G x$$

is the unique morphism of Banach–Lie groups $(\mathfrak{u}, *) \to G$ integrating the inclusion map $\mathfrak{u} \to \mathfrak{g}$. This implies that $U = \exp_G(\mathfrak{u})$ is a subgroup of G.

To see that it is a Lie subgroup, we recall that P is a Lie subgroup of G (Definition 1.11). This implies that

$$U_1 := \{p \in P \colon (\mathrm{Ad}(p) - \mathbf{1})(\mathfrak{p}) \subseteq \mathfrak{u}\}$$

also is a Lie subgroup of P and hence of G (use Lemma IV.11 in [Ne04] and generalize Lemma IV.12 slightly). Clearly,

$$\mathbf{L}(U_1) = \{x \in \mathfrak{p} \colon [x, \mathfrak{p}] \subseteq \mathfrak{u}\} \supseteq \mathfrak{u},$$

but, conversely, $[x, \mathfrak{h}] \subseteq \mathfrak{u}$ implies $x \in \mathfrak{u}$, so that $\mathbf{L}(U_1) = \mathfrak{u}$. We conclude that U is the identity component of the Lie subgroup U_1, hence a Lie subgroup.

Finally, we show that $\exp_U$ is a diffeomorphism. Since U is connected and nilpotent, it suffices to show that $\exp_U$ is injective. So let $x \in \mathfrak{u}$ with $\exp_G x = \mathbf{1}$. Then $\mathbf{1} = \mathrm{Ad}(\exp_G x) = e^{\mathrm{ad}\, x}$, and since $\mathrm{ad}\, x$ is nilpotent, we get $\mathrm{ad}\, x = 0$. Since the root decomposition yields $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{g}_0$, we see that $x = 0$.

(b) follows from (a), applied to the parabolic subalgebra defined by $-\Sigma$.

(c) That L is a Lie subgroup follows from [Ne04, Lemmas IV.11/12]. In view of Lemma 1.9, its Lie algebra is $\mathfrak{p} \cap \mathfrak{n}_\mathfrak{g}(\mathfrak{l}) \cap \mathfrak{n}_\mathfrak{g}(\mathfrak{n}) = \mathfrak{l}$.

Since L normalizes $\mathfrak{u}$ and $\mathfrak{n}$, its acts by the adjoint action on these Lie algebras, and (a) and (b) imply that this corresponds to a holomorphic action by conjugation on the groups U and N.

(d) Since L acts holomorphically by conjugation on U, the semidirect product $U \rtimes L$ is a complex Banach–Lie group. As $\mathfrak{p} = \mathfrak{u} \rtimes \mathfrak{l}$ is a semidirect sum, the canonical homomorphism $U \rtimes L \to P, (u, l) \mapsto ul$ is a morphism of Banach–Lie groups which is a local diffeomorphism. Therefore UL is an open subgroup of P and the multiplication map $U \times L \to UL$ is a covering. It remains to see that this map is injective, i.e., that $U \cap L = \{\mathbf{1}\}$.

Let $x \in \mathfrak{u}$. If $\exp_G x \in L$, then the unipotent operator $e^{\mathrm{ad}\, x}$ normalizes $\mathfrak{l}$, and this implies that $\mathrm{ad}\, x = \log(e^{\mathrm{ad}\, x})$ also preserves $\mathfrak{l}$. Hence $x \in \mathfrak{n}_\mathfrak{g}(\mathfrak{l}) = \mathfrak{l}$ leads to $x = 0$.

(e) The direct product Lie group $N \times (U \rtimes L)$ acts smoothly on G by

$$(n, (u, l)).g := ngl^{-1}u^{-1},$$

so that (e) means that the orbit map of $\mathbf{1}$ is a diffeomorphism onto an open subset. That the differential in $\mathbf{1}$ is a linear isomorphism follows from equation (3),

so that the orbit map is a local diffeomorphism by the Inverse Function Theorem, hence a covering map whose range is an open subset of G. It remains to verify that the orbit map is injective, i.e., $P \cap N = \{\mathbf{1}\}$. This can be seen as in (d): If the ad-nilpotent element $x \in \mathfrak{n}$ satisfies $e^{\operatorname{ad} x}\mathfrak{p} = \mathfrak{p}$, then $x \in \mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}) = \mathfrak{p}$ yields $x = 0$, and this prove that $N \cap P = \{\mathbf{1}\}$. ∎

2   Coinduced Banach representations

In this section we discuss coinduced representation of Banach–Lie algebras. After introducing a suitable notion of continuity for linear maps $\alpha \colon U(\mathfrak{g}) \to E$ on the enveloping algebra $U(\mathfrak{g})$ of a Banach–Lie algebra $\mathfrak{g}$ with values in a Banach space E, we show that the space $\operatorname{Hom}(U(\mathfrak{g}), E)_c$ of all these maps carries a natural Fréchet space structure for which the left and right action of $\mathfrak{g}$ are continuous. If $\mathfrak{p} \subseteq \mathfrak{g}$ is a complemented closed subalgebra, this leads for each continuous representation $\rho \colon \mathfrak{p} \to \mathfrak{gl}(E)$ to a Fréchet structure on the corresponding coinduced representation $\operatorname{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)_c$. Specializing all that to a parabolic subalgebra $\mathfrak{p}$ of a weakly root graded Lie algebra $\mathfrak{g}$, our main result is that, provided E decomposes into finitely many $\mathfrak{h}$-weight spaces, the subspace $\operatorname{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)_c^{[\mathfrak{g}_\triangle]}$ of $\mathfrak{g}_\triangle$-finite elements is closed and a Banach space (Theorem 2.10). As we shall see in the next section, this implies that on the group level, holomorphic induction from Banach representations of P yields Banach representations of G.

Coinduced Fréchet representations

Definition 2.1 Let V and W be Banach spaces. For an n-linear map $m \colon V^n \to W$, we define

$$\|m\| := \sup\{\|m(v_1, \ldots, v_n)\| \colon v_1, \ldots, v_n \in V, \|v_i\| \leq 1\} \in [0, \infty].$$

Then m is continuous if and only if $\|m\| < \infty$, and we have

$$\|m(v_1, \ldots, v_n)\| \leq \|m\| \|v_1\| \cdots \|v_n\|, \quad v_i \in V.$$

We thus obtain on the space $\operatorname{Mult}^n(V, W)$ of continuous W-valued n-linear maps on V^n the structure of a Banach space.

Definition 2.2 Let $\mathfrak{g}$ be a Banach–Lie algebra and E be a Banach space. We call a linear map $\beta \colon U(\mathfrak{g}) \to E$ *continuous* if for each $n \in \mathbb{N}_0$, the n-linear map

$$\beta_n \colon \mathfrak{g}^n \to E, \quad (x_1, \ldots, x_n) \mapsto \beta(x_1 \cdots x_n)$$

is continuous. We write $\operatorname{Hom}(U(\mathfrak{g}), E)_c$ for the set of all continuous linear maps and define a family of seminorms

$$p_n \colon \operatorname{Hom}(U(\mathfrak{g}), E)_c \to \mathbb{R}, \quad \beta \mapsto \|\beta_n\|.$$

Lemma 2.3 *With respect to the sequence $(p_n)_{n \in \mathbb{N}_0}$ of seminorms, the space $\mathrm{Hom}(U(\mathfrak{g}), E)_c$ is a Fréchet space for which all evaluation maps*

$$\mathrm{ev}_D \colon \mathrm{Hom}(U(\mathfrak{g}), E)_c \to E, \quad \beta \mapsto \beta(D)$$

are continuous. Moreover, the left representation of $U(\mathfrak{g})$ on this space defined by

$$\mathfrak{g} \times \mathrm{Hom}(U(\mathfrak{g}), E)_c \to \mathrm{Hom}(U(\mathfrak{g}), E)_c, \quad (x, \beta) \mapsto -\beta \circ \lambda_x, \quad \lambda_x(D) = xD,$$

and the right representation

$$\mathfrak{g} \times \mathrm{Hom}(U(\mathfrak{g}), E)_c \to \mathrm{Hom}(U(\mathfrak{g}), E)_c, \quad (x, \beta) \mapsto \beta \circ \rho_x, \quad \rho_x(D) = Dx,$$

are continuous bilinear maps.

Proof. Since the canonical map $\mathrm{Hom}(U(\mathfrak{g}), E)_c \to \prod_{n \in \mathbb{N}_0} \mathrm{Mult}^n(\mathfrak{g}, E)$ is injective, the sequence of seminorms $(p_n)_{n \in \mathbb{N}_0}$ defines on $\mathrm{Hom}(U(\mathfrak{g}), E)_c$ a metrizable locally convex topology. It also follows immediately from the definitions, that all evaluation maps ev_D are continuous because we may write D as a finite sum $D = D_0 + \ldots + D_k$, where D_j is a j-fold product $x_1 \cdots x_j$ of elements $x_i \in \mathfrak{g}$.

To verify that $\mathrm{Hom}(U(\mathfrak{g}), E)_c$ is complete, i.e., a Fréchet space, let $(\beta^{(n)})_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathrm{Hom}(U(\mathfrak{g}), E)_c$. Then, for each $k \in \mathbb{N}_0$, the sequence $(\beta_k^{(n)})_{n \in \mathbb{N}}$ define a Cauchy sequence in the Banach space $\mathrm{Mult}^k(\mathfrak{g}, E)$, which converges to some limit β_k. We also note that for each $D \in U(\mathfrak{g})$, the sequence $(\beta^{(n)}(D))_{n \in \mathbb{N}}$ in E is Cauchy, hence convergent. If we write $\beta \colon U(\mathfrak{g}) \to E$ for the pointwise limit, we thus obtain a linear map whose corresponding k-linear maps $\mathfrak{g}^k \to E$ coincide with the continuous maps β_k. This implies that $\beta \in \mathrm{Hom}(U(\mathfrak{g}), E)_c$ and that $\beta^{(n)} \to \beta$ holds in the metric topology on $\mathrm{Hom}(U(\mathfrak{g}), E)_c$.

Next we observe that for $\beta \in \mathrm{Hom}(U(\mathfrak{g}), E)_c$ and $x \in \mathfrak{g}$ we have

$$p_n(\beta \circ \lambda_x) \leq \|x\| \cdot p_{n+1}(\beta) \quad \text{and} \quad p_n(\beta \circ \rho_x) \leq \|x\| \cdot p_{n+1}(\beta). \tag{4}$$

This proves that the corresponding bilinear maps

$$\mathfrak{g} \times \mathrm{Hom}(U(\mathfrak{g}), E)_c \to \mathrm{Mult}^n(\mathfrak{g}, E), \quad (x, \beta) \mapsto (\beta \circ \lambda_x)_n, (\beta \circ \rho_x)_n,$$

are continuous. Since the topology on $\mathrm{Hom}(U(\mathfrak{g}), E)_c$ is obtained by the embedding into the product of the spaces $\mathrm{Mult}^n(\mathfrak{g}, E)$, the assertion follows. ∎

Next we assume that $\mathfrak{p}$ and $\mathfrak{n}$ are closed subalgebras of $\mathfrak{g}$ with $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{p}$ (topological direct sum) and that (ρ_E, E) is a continuous representation of $\mathfrak{p}$ on the Banach space E.

Definition 2.4 For a continuous morphism $\rho_E \colon \mathfrak{p} \to \mathfrak{gl}(E)$ of Banach–Lie algebras, we consider the subspace

$$\mathrm{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)_c := \{\beta \in \mathrm{Hom}(U(\mathfrak{g}), E)_c \colon (\forall x \in \mathfrak{p}) \, \beta \circ \rho_x = -\rho_E(x) \circ \beta\}.$$

From Lemma 2.3 we immediately derive that this is a closed subspace of the Fréchet space $\operatorname{Hom}(U(\mathfrak{g}), E)_c$, hence also a Fréchet space, and that the left action of $\mathfrak{g}$ on this space is continuous. It is called the *coinduced representation* defined by (ρ_E, E).

Since the multiplication map $U(\mathfrak{n}) \otimes U(\mathfrak{p}) \to U(\mathfrak{g})$ is a linear isomorphism by the Poincaré–Birkhoff–Witt Theorem, the restriction map

$$R \colon \operatorname{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)_c \cong \operatorname{Hom}(U(\mathfrak{n}), E)_c$$

is injective. We even have more:

Lemma 2.5 R *is a topological isomorphism of Fréchet spaces.*

Proof.   Clearly, R is injective, and the PBW Theorem implies that it is bijective on the algebraic level. Each $\alpha \in \operatorname{Hom}(U(\mathfrak{n}), E)$ extends by

$$\widetilde{\alpha}(D \cdot x_1 \cdots x_n) := (-1)^n \rho_E(x_n) \cdots \rho_E(x_1) \alpha(D), \quad x_i \in \mathfrak{p}, D \in U(\mathfrak{n}),$$

to a unique linear map $\widetilde{\alpha} \in \operatorname{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)$. It remains to show that the associated n-linear maps $\widetilde{\alpha}_n$ are continuous if all maps α_n are continuous.

To this end, we show that there exist constants $C_{j,n}$, $j \leq n$, such that

$$p_n(\widetilde{\alpha}) \leq \sum_{j \leq n} C_{j,n} p_j(\alpha). \tag{5}$$

To verify these estimates, let $C_{\mathfrak{g}} \geq 0$ be such that

$$\|[x, y]\| \leq C_{\mathfrak{g}} \|x\| \cdot \|y\| \quad \text{for} \quad x, y \in \mathfrak{g},$$

and assume w.l.o.g. that

$$\|x + y\| = \max(\|x\|, \|y\|) \quad \text{for} \quad x \in \mathfrak{p}, y \in \mathfrak{n}.$$

We now prove (5) by induction on n. For $n = 0$, we only deal with the constant term, so that $p_0(\widetilde{\alpha}) = p_0(\alpha)$. Next we assume that $n > 0$ and let $z_i = x_i + y_i \in \mathfrak{g}$, $x_i \in \mathfrak{p}$, $y_i \in \mathfrak{n}$, $i = 1, \ldots, n$ with $\|z_i\| \leq 1$. Then

$$\begin{aligned}
\widetilde{\alpha}(z_1 \cdots z_n) &= \widetilde{\alpha}(z_1 \cdots z_{n-1} \cdot (x_n + y_n)) \\
&= -\rho_E(x_n) \widetilde{\alpha}(z_1 \cdots z_{n-1}) - \widetilde{\alpha}([y_n, z_1 \cdots z_{n-1}]) + \widetilde{\alpha}(y_n \cdot z_1 \cdots z_{n-1}) \\
&= -\rho_E(x_n) \widetilde{\alpha}(z_1 \cdots z_{n-1}) - \widetilde{\alpha}([y_n, z_1 \cdots z_{n-1}]) + (\alpha \circ \lambda_{y_n})\widetilde{}(z_1 \cdots z_{n-1}).
\end{aligned}$$

For the last term we obtain with the induction hypothesis and (4) the estimate

$$\|(\alpha \circ \lambda_{y_n})\widetilde{}(z_1 \cdots z_{n-1})\| \leq p_{n-1}\big((\alpha \circ \lambda_{y_n})\widetilde{}\big) \leq \sum_{j \leq n-1} C_{j,n-1} p_{j+1}(\alpha).$$

For the other two terms, we have

$$\begin{aligned}
\|\rho_E(x_n) \widetilde{\alpha}(z_1 \cdots z_{n-1})\| &\leq p_{n-1}(\rho_E(x_n) \circ \widetilde{\alpha}) \leq \|\rho_E\| p_{n-1}(\widetilde{\alpha}) \\
&\leq \|\rho_E\| \sum_{j \leq n-1} C_{j,n-1} p_j(\alpha)
\end{aligned}$$

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and $[y_n, z_1 \cdots z_{n-1}] = \sum_{i=1}^{n-1} z_1 \cdots z_{i-1}[y_n, z_i]z_{i+1} \cdots z_{n-1}$ leads to

$$\|\widetilde{\alpha}([y_n, z_1 \cdots z_{n-1}])\| \le (n-1)C_{\mathfrak{g}} p_{n-1}(\widetilde{\alpha}) \le (n-1)C_{\mathfrak{g}} \sum_{j \le n-1} C_{j,n-1} p_j(\alpha).$$

This completes the inductive proof of (5). ∎

Remark 2.6 Let $k \in \mathbb{N}_0$ and consider in $\operatorname{Hom}(U(\mathfrak{n}), E)_c$ the subspace

$$\operatorname{Hom}(U(\mathfrak{n}), E)_c^k := \{\alpha \in \operatorname{Hom}(U(\mathfrak{n}), E)_c \colon (\forall n \ge k) \, \alpha_n = 0\}.$$

This is a closed subspace of the Fréchet space $\operatorname{Hom}(U(\mathfrak{n}), E)_c$, and since all but finitely many of the seminorms p_n vanish on this space, it actually is a Banach space.

Applications to weakly root graded Lie algebras

In this subsection we return to the setting where $\mathfrak{p}$ is a parabolic subalgebra of the weakly root graded complex Banach–Lie algebra $\mathfrak{g}$.

Lemma 2.7 *For a $\mathfrak{g}$-module V with an $\mathfrak{h}$-weight decomposition, the following are equivalent:*

(a) *The set of $\mathfrak{h}$-weights of V is finite.*

(b) *V is a locally finite $\mathfrak{g}_\triangle$-module with finitely many isotypic components.*

Then V is a semisimple $\mathfrak{g}_\triangle$-module, hence in particular a direct sum of simple ones.

Proof. (a) implies that the $\mathfrak{g}_\triangle$-module V is integrable in the sense that each root vector $x_\alpha \in \mathfrak{g}_{\triangle, \alpha}$ acts as a nilpotent operator on V. Hence [Ne03, Thm. A.1(1), Prop. A.2] imply that V is a locally finite (and therefore semisimple) $\mathfrak{g}_\triangle$-module with finitely many isotypic components. That (b) implies (a) is trivial. ∎

Lemma 2.8 *Let $\mathfrak{k} \subseteq \mathfrak{g}$ be a subalgebra for which $\mathfrak{g}$ is a locally finite $\mathfrak{k}$-module. Then for any representation (π, V) of $\mathfrak{g}$, the subspace*

$$V^{[\mathfrak{k}]} := \{v \in V \colon \dim\big(U(\mathfrak{k})v\big) < \infty\}$$

of $\mathfrak{k}$-finite vectors is a $\mathfrak{g}$-submodule.

Proof. Since $\mathfrak{g}$ is locally $\mathfrak{k}$-finite, the tensor algebra over $\mathfrak{g}$, and hence $U(\mathfrak{g})$, is a locally $\mathfrak{k}$-finite $\mathfrak{g}$-module. This property carries over to the tensor product $U(\mathfrak{g}) \otimes V^{[\mathfrak{k}]}$, and hence to its image under the $\mathfrak{k}$-equivariant evaluation map $U(\mathfrak{g}) \otimes V^{[\mathfrak{k}]} \to V$. We conclude that $U(\mathfrak{g})V^{[\mathfrak{k}]} \subseteq V^{[\mathfrak{k}]}$. ∎

Definition 2.9 We define the *Weyl group* $\mathcal{W} := \mathcal{W}(\mathfrak{g}_\Delta, \mathfrak{h}) \subseteq \mathrm{GL}(\mathfrak{h}^*)$ as the subgroup generated by the reflections $r_\alpha(\beta) := \beta - \beta(\check{\alpha})\alpha$.

Theorem 2.10 (Finiteness Theorem) *Let $\mathfrak{p} \subseteq \mathfrak{g}$ be a parabolic subalgebra and (ρ, E) be a $\mathfrak{p}$-module decomposing into a finite sum of $\mathfrak{h}$-weight spaces and $V := \mathrm{Hom}_\mathfrak{p}(U(\mathfrak{g}), E)^{[\mathfrak{h}]}$ be the $\mathfrak{h}$-finite part of the corresponding coinduced representation. Then the following assertions hold:*

(a) *V is a direct sum of $\mathfrak{h}$-weight spaces and the set $\mathcal{P}(V, \mathfrak{h})$ of $\mathfrak{h}$-weights in V is contained in $\mathcal{P}(E, \mathfrak{h}) - \mathrm{span}_{\mathbb{N}_0} \Sigma^-$.*

(b) *The $\mathfrak{g}_\Delta$-finite subspace $V^{[\mathfrak{g}_\Delta]}$ is a $\mathfrak{g}$-module decomposing into finitely many $\mathfrak{h}$-weight spaces.*

(c) *V is a locally finite module of $\mathfrak{n}_\Delta \rtimes \mathfrak{l}_\Delta$.*

(d) *There exists a $k \in \mathbb{N}$ with $V^{[\mathfrak{g}_\Delta]} = \{v \in V \colon \mathfrak{u}_\Delta^k.v = \{0\}\}$.*

(e) *There exists a $k \in \mathbb{N}$ with $V^{[\mathfrak{g}_\Delta]} \subseteq \{v \in V \colon \mathfrak{n}^k.v = \{0\}\}$.*

Proof.   (a) First we recall that the multiplication map $U(\mathfrak{n}) \otimes U(\mathfrak{p}) \to U(\mathfrak{g})$ is a linear isomorphism (Poincaré–Birkhoff–Witt), so that we obtain a linear isomorphism $\mathrm{Hom}_\mathfrak{p}(U(\mathfrak{g}), E) \cong \mathrm{Hom}(U(\mathfrak{n}), E)$, of $\mathfrak{h}$-modules. If $E = \bigoplus_{\alpha \in \mathcal{P}(E, \mathfrak{h})} E_\alpha$ is the finite weight space decomposition of E, we accordingly obtain a product decomposition of $\mathfrak{h}$-modules

$$\mathrm{Hom}(U(\mathfrak{n}), E) \cong \prod_{\alpha \in \mathcal{P}(E, \mathfrak{h})} \mathrm{Hom}(U(\mathfrak{n}), E_\alpha).$$

Since $\mathfrak{n}$ is a finite sum of $\mathfrak{h}$-weight spaces, the enveloping algebra $U(\mathfrak{n})$ is an (infinite) direct sum of $\mathfrak{h}$-weight spaces $U(\mathfrak{n})_\beta$, which leads to a product decomposition of $\mathfrak{h}$-modules

$$\mathrm{Hom}(U(\mathfrak{n}), E_\alpha) \cong \prod_\beta \mathrm{Hom}(U(\mathfrak{n})_\beta, E_\alpha),$$

and $\mathrm{Hom}(U(\mathfrak{n})_\beta, E_\alpha) \subseteq V_{\alpha - \beta}$. Therefore the set $\mathcal{P}(V, \beta)$ of $\mathfrak{h}$-weights on V is given by

$$\mathcal{P}(V, \mathfrak{h}) = \mathcal{P}(E, \mathfrak{h}) - \mathrm{span}_{\mathbb{N}_0} \Sigma^-,$$

and $V = \sum_\alpha \sum_\beta \mathrm{Hom}(U(\mathfrak{n})_\beta, E_\alpha)$ follows from the fact that an element of $\mathrm{Hom}(U(\mathfrak{n}), E_\alpha)$ is $\mathfrak{h}$-finite if and only if only finitely many components in the spaces $\mathrm{Hom}(U(\mathfrak{n})_\beta, E_\alpha)$ are non-zero.

(b) Since $\mathfrak{g}$ is a locally finite module of the subalgebras $\mathfrak{h}$ and $\mathfrak{g}_\Delta$, Lemma 2.8 implies that V is a $\mathfrak{g}$-submodule of $\mathrm{Hom}_\mathfrak{p}(U(\mathfrak{g}), E)$ and that $V^{[\mathfrak{g}_\Delta]}$ is a $\mathfrak{g}$-submodule of V.

Let $\mathcal{R}^+$ be a positive system contained in $-\Sigma$ (Definition 1.4), so that we necessarily have $\Sigma^- \subseteq \mathcal{R}^+$. We have seen in (a) that

$$\mathcal{P}(V, \mathfrak{h}) \subseteq \mathcal{P}(E, \mathfrak{h}) - \mathrm{span}_{\mathbb{N}_0} \Sigma^- \subseteq \mathcal{P}(E, \mathfrak{h}) - C, \tag{6}$$

where $C := \mathrm{span}_{\mathbb{R}_+} \mathcal{R}^+$ is a polyhedral convex cone containing no non-zero linear subspace.

The local finiteness of the $\mathfrak{g}_\Delta$-module $V^{[\mathfrak{g}_\Delta]}$ implies that the set $\mathcal{P}_f := \mathcal{P}(V^{[\mathfrak{g}_\Delta]}, \mathfrak{h})$ is invariant under the Weyl group $\mathcal{W}$ ([MP95]), which leads to

$$\mathcal{P}(V^{[\mathfrak{g}_\Delta]}, \mathfrak{h}) \subseteq D := \bigcap_{w \in \mathcal{W}} w(\mathrm{conv}(\mathcal{P}(E, \mathfrak{h})) - C). \tag{7}$$

The set D is convex and we claim that it is compact. If this is not the case, then there exists a non-zero $\beta \in \mathfrak{h}_{\mathbb{R}}^*$ with $D + \mathbb{R}_+\beta \subseteq D$ ([Ne00, Prop. V.1.6]). Pick $d \in D$. Then $d + \mathbb{R}_+\beta \subseteq w(\mathrm{conv}(\mathcal{P}(E, \mathfrak{h})) - C)$ implies that $\beta \in -wC$ for each $w \in \mathcal{W}$ ([Ne00, Prop. V.1.6]). We thus arrive at $\mathcal{W}\beta \subseteq -C$. By definition of a positive system, there exists an element $x \in \mathfrak{h}$ with $\mathcal{R}^+(x) > 0$, and then $\gamma(x) > 0$ holds for each non-zero element $\gamma \in C$, in particular $(w\beta)(x) < 0$ for each $w \in \mathcal{W}$ and for $\gamma := \sum_{w \in \mathcal{W}} w\beta$ we also obtain $\gamma(x) < 0$. Since γ is $\mathcal{W}$-invariant, it vanishes on all coroots, which leads to the contradiction $\gamma = 0$. We conclude that

$$\bigcap_{w \in W} wC = \{0\}. \tag{8}$$

This proves that D is compact. Now the discreteness of the set $\mathcal{P}(E, \mathfrak{h}) + \mathrm{span}_{\mathbb{Z}} \mathcal{R}$ in $\mathfrak{h}_{\mathbb{R}}^*$ implies that $\mathcal{P}_f$ is finite. As $\mathfrak{h}$ is diagonalizable on V, (b) follows.

(c) In view of the PBW Theorem, it suffices to show that V is a locally finite module of $\mathfrak{n}_\Delta$ and $\mathfrak{l}_\Delta$.

To see that it is locally finite for $\mathfrak{n}_\Delta$, we pick $x \in \mathfrak{h}$ with $\Sigma^0(x) = \{0\}$ and $\Sigma^-(x) \geq 1$ (cf. Remark 1.5). Then (a) implies that $\mathcal{P}(V, \mathfrak{h})(x)$ is bounded from above. Therefore $\mathfrak{g}_\alpha V_\beta \subseteq V_{\alpha+\beta}$ and $\Sigma^-(x) \geq 1$ imply that for each weight β of V, there exists a $k \in \mathbb{N}$ with $V_{\beta+\alpha_1+\ldots+\alpha_k} = \{0\}$ for $\alpha_i \in \Sigma^-$. We thus derive that V is a locally nilpotent $\mathfrak{n}$-module, hence in particular locally finite for the finite dimensional Lie algebra $\mathfrak{n}_\Delta$.

Next we note that on each space $\mathrm{Hom}(U(\mathfrak{n})_\beta, E_\alpha)$, the eigenvalue of $x \in \mathfrak{h}$ is given by $\alpha(x) - \beta(x)$ and, for each $c \in \mathbb{R}$, the set $\{\beta \in \mathrm{span}_{\mathbb{N}_0} \Sigma^- : \beta(x) = c\}$ is finite. Since x commutes with $\mathfrak{l}$, the eigenspaces of x in V are $\mathfrak{l}$-invariant, and the preceding argument shows that they decompose into finitely many $\mathfrak{h}$-weight spaces. Therefore Lemma 2.7 implies that each such eigenspace is a locally finite $\mathfrak{l}_\Delta$-module. Here we use Remark 1.8 to see that $\mathfrak{l}$ is Δ^0-graded, so that Lemma 2.7 applies.

(d) In view of (c) and $U(\mathfrak{g}_\Delta) = U(\mathfrak{u}_\Delta)U(\mathfrak{l}_\Delta)U(\mathfrak{n}_\Delta)$, the subspace $V^{[\mathfrak{g}_\Delta]}$ of $\mathfrak{g}_\Delta$-finite elements coincides with the subspace $V^{[\mathfrak{u}_\Delta]}$ of $\mathfrak{u}_\Delta$-finite elements. Further, the finiteness of $\mathcal{P}_f$ and $\Sigma^+(x) \leq -1$ entail the existence of some $k \in \mathbb{N}$ with

$$V^{[\mathfrak{g}_\Delta]} \subseteq \{v \in V : (\mathfrak{u}_\Delta)^k.v = \{0\}\}.$$

Conversely, the condition $(\mathfrak{u}_\Delta)^k.v = \{0\}$ obviously implies that v is $\mathfrak{u}_\Delta$-finite and hence $\mathfrak{g}_\Delta$-finite by (c). We therefore obtain the desired equality (d).

(e) follows from the finiteness of $\mathcal{P}_f$ and $\Sigma^-(x) \geq 1$. $\blacksquare$

Remark 2.11 Equations (7) and (8) have an interesting consequence. If E is a trivial $\mathfrak{p}$-module, then $\mathcal{P}(E, \mathfrak{h}) = \{0\}$ implies that

$$\mathcal{P}(V^{[\mathfrak{g}_\Delta]}, \mathfrak{h}) \subseteq - \bigcap_{w \in \mathcal{W}} wC = \{0\},$$

so that $\mathfrak{h}$ acts trivially on $V^{[\mathfrak{g}_\Delta]}$. This in turn implies that all root spaces $\mathfrak{g}_\alpha$, $\alpha \in \Delta$, act trivially on $V^{[\mathfrak{g}_\Delta]}$. If $\mathfrak{g}$ is root graded, we conclude that the $\mathfrak{g}$-module $V^{[\mathfrak{g}_\Delta]}$ is trivial, and we find that

$$V^{[\mathfrak{g}_\Delta]} = \mathrm{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)^{\mathfrak{g}} \cong E.$$

The preceding theorem is of an algebraic nature, but it has some interesting consequences for Banach representations.

Theorem 2.12 (Coinduced Banach representations) *Let (ρ_E, E) be a Banach representation of $\mathfrak{p}$ which decomposes into finitely many $\mathfrak{h}$-weight spaces and endow $V_c := \mathrm{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)_c$ with its natural Fréchet structure. Then the subspace $V_f := V_c^{[\mathfrak{g}_\Delta]}$ of $\mathfrak{g}_\Delta$-finite elements in V_c is a closed $\mathfrak{g}$-invariant subspace which is Banach and the representation of $\mathfrak{g}$ on this space is continuous.*

Proof. First we apply Theorem 2.10 to the subspace V_c of $\mathrm{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)$. Since each $\mathfrak{g}_\Delta$-finite vector is in particular $\mathfrak{h}$-finite, $V_f \subseteq \mathrm{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)^{[\mathfrak{h}]}$, so that Theorem 2.10(d) implies the existence of a $k \in \mathbb{N}$ with

$$V_f = \{v \in V_c \colon (\mathfrak{u}_\Delta^k).v = \{0\}\}.$$

This implies that V_f is a closed subspace of the Fréchet space V_c. Next we use Theorem 2.10(e) to find a $k \in \mathbb{N}$ with $\mathfrak{n}^k.V_f = \{0\}$. Then, for each $\alpha \in V_f$ and $m \in \mathbb{N}$, the relation

$$(\mathfrak{n}^k.\alpha)(\mathfrak{n}^m) = \alpha(\mathfrak{n}^{k+m}) = \{0\}$$

implies that

$$V_f|_{U(\mathfrak{n})} \subseteq \mathrm{Hom}(U(\mathfrak{n}), E)_c^k,$$

and since the right hand side is a Banach space (Remark 2.6), Lemma 2.5 implies that V_f is a Banach space. The continuity of the $\mathfrak{g}$-action on the Banach space V_f is inherited from the continuity on the Fréchet space V_c. ∎

We now turn to some additional information on the $\mathfrak{g}$-representation on V_f.

Proposition 2.13 *Let (ρ_E, E) be an irreducible Banach representation of $\mathfrak{p}$ which decomposes into finitely many $\mathfrak{h}$-weight spaces and assume that V_f is non-zero. Then the following assertions hold:*

(i) *$\mathfrak{u}$ acts trivially on E.*

(ii) *Each closed $\mathfrak{g}$-submodule of the Banach space V_f contains the space $E \cong \mathrm{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)_c^{\mathfrak{n}}$ of $\mathfrak{n}$-invariants. In particular, E generates a unique closed minimal submodule which is irreducible.*

(iii) *For each weight $\alpha \in \mathcal{P}(E, \mathfrak{h})$, we have*

$$V_{f,\alpha} = E_\alpha \quad and \quad \alpha(\check{\beta}) \in \mathbb{N}_0 \quad for \quad \beta \in \Sigma^- \cap \Delta.$$

If, in addition, $\dim E = 1$ and $\lambda := \rho|_{\mathfrak{h}}$, then the weight space $V_{f,\lambda}$ is one dimensional.

(iv) *For each $\mathfrak{p}$-equivariant continuous map $\varphi \colon V_f \to E$, there exists a unique $\psi \in \operatorname{End}_{\mathfrak{p}}(E)$ with $\varphi = \psi \circ \operatorname{ev}_{\mathbf{1}}$.*

Proof. (i) Since E decomposes into finitely many $\mathfrak{h}$-weight spaces, we have $\mathfrak{u}^k.E = \{0\}$ for some k. In particular, $E^{\mathfrak{u}}$ is a non-zero closed $\mathfrak{p}$-submodule. Since E was assumed to be irreducible, it follows that $\mathfrak{u} \subseteq \ker \rho_E$, and hence that E is an irreducible $\mathfrak{l}$-module.

(ii) We have already seen in Theorem 2.10(e) that V_f is a nilpotent $\mathfrak{n}$-module, and this property is inherited by any $\mathfrak{g}$-submodule $W \subseteq V_f$. Hence $W^{\mathfrak{n}} \neq \{0\}$ whenever $W \neq \{0\}$. For the $\mathfrak{n}$-invariant vectors we observe that

$$\operatorname{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)_c^{\mathfrak{n}} \cong \operatorname{Hom}(U(\mathfrak{n}), E)_c^{\mathfrak{n}} \cong \operatorname{Hom}(U(\mathfrak{n})/\mathfrak{n}U(\mathfrak{n}), E)_c \cong E.$$

In this sense we identify E with the $\mathfrak{l}$-submodule $\operatorname{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)_c^{\mathfrak{n}}$ of $\operatorname{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)_c$. As $W^{\mathfrak{n}}$ is a non-zero closed $\mathfrak{l}$-submodule of E, we thus obtain $W^{\mathfrak{n}} = E$ and therefore $E \subseteq W$.

(iii) Next we pick $x_\Sigma \in \mathfrak{h}$ as in Remark 1.5. Then x_Σ is central in $\mathfrak{l}$ and $\rho_E(x_\Sigma)$ is diagonalizable. Since all eigenspaces of this element are $\mathfrak{l}$-submodules, the irreducibility of E entails that $\rho_E(x_\sigma) = c\operatorname{id}_E$ for some $c \in \mathbb{C}$. On the other hand, $\Sigma^-(x_\Sigma) \leq -1$, and $U(\mathfrak{n})$ decomposes into eigenspaces $U(\mathfrak{n})_\mu$ of x_Σ with $\mu \in\,]-\infty, -1]$, where $U(\mathfrak{n})_0 = \mathbb{C}\mathbf{1}$. Therefore

$$V_f \subseteq \bigoplus_\mu \operatorname{Hom}(U(\mathfrak{n})_\mu, E),$$

and the eigenvalue of x_Σ on $\operatorname{Hom}(U(\mathfrak{n})_\mu, E)$ is $c - \mu$, which is different from c if $\mu \neq 0$. This proves that

$$V_{f,c}(x_\Sigma) = \{v \in V_f \colon x_\Sigma v = cv\} = E,$$

and therefore $V_{f,\alpha} = E_\alpha$ for each $\mathfrak{h}$-weight α of E.

Let $\beta \in \Delta \cap \Sigma^-$ and $\mathfrak{g}_\Delta(\beta) := \mathfrak{g}_{\Delta,\beta} + \mathfrak{g}_{\Delta,-\beta} + \mathbb{C}\check{\beta} \cong \mathfrak{sl}_2(\mathbb{C})$ be the $\mathfrak{sl}_2$-subalgebra corresponding to β. Then the preceding argument shows that each element of E_α generates a finite dimensional $\mathfrak{g}_\Delta(\beta)$-module for which the $\check{\beta}$-eigenvalues are contained in $\alpha(\check{\beta}) - 2\mathbb{N}_0$, so that $\mathfrak{sl}_2$-theory yields $\alpha(\check{\beta}) \in \mathbb{N}_0$.

(iv) Since $\mathfrak{h} \subseteq \mathfrak{p}$, φ annihilates each weight space $V_{f,\beta}$ with $\beta \notin \mathcal{P}(E, \mathfrak{h})$. In view of (iii), the fact that V_f is a direct sum of $\mathfrak{h}$-weight spaces implies that

$$V_f \cong E \oplus \sum_{\beta \notin \mathcal{P}(E,\mathfrak{h})} V_{f,\beta},$$

and by identifying $\operatorname{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)_c$ with $\operatorname{Hom}(U(\mathfrak{n}), E)_c$ (Lemma 2.5), we see that $\ker(\operatorname{ev}_{\mathbf{1}})$ coincides with the sum of all $\mathfrak{h}$-weight spaces $V_{f,\beta}$, $\beta \notin \mathcal{P}(E, \mathfrak{h})$. We conclude that φ vanishes on $\ker(\operatorname{ev}_{\mathbf{1}})$, hence induces a unique continuous $\mathfrak{p}$-equivariant map $\psi \in \operatorname{End}_{\mathfrak{p}}(E)$ with $\psi \circ \operatorname{ev}_{\mathbf{1}} = \varphi$. ∎

3   Holomorphically induced representations

Let G be a weakly root graded complex Banach–Lie group. Each closed subalgebra of $\mathfrak{g} := \mathrm{L}(G)$ integrates to an integral subgroup (cf. [GN09]). In particular, we obtain an integral subgroup G_Δ corresponding to the inclusion $\mathfrak{g}_\Delta \hookrightarrow \mathfrak{g}$. We write H for the Cartan subgroup of G_Δ corresponding to the Cartan subalgebra $\mathfrak{h}$ and $T \subseteq H$ for its (unique) maximal compact subgroup, a real torus. For $r := \dim T$, we have $T \cong \mathbb{T}^r$, $H \cong (\mathbb{C}^\times)^r$, and $H \cong T_{\mathbb{C}}$ is the universal complexification of T, so that we may identify the group $\widehat{T} = \mathrm{Hom}(T, \mathbb{T})$ of continuous characters of T with the group $\widehat{H} := \mathrm{Hom}_{\mathcal{O}}(H, \mathbb{C}^\times)$ of holomorphic characters of H.

Theorem 3.1 (Peter–Weyl) *If $\pi\colon K \to \mathrm{GL}(V)$ is a homomorphism defining a continuous linear action of the compact group K on the complete locally convex space V, then the following assertions hold:*

(1) *The space $V^{[K]}$ of K-finite vectors is dense in V.*

(2) *For each irreducible representation $[\chi] \in \widehat{K}$, there is a continuous projection $p_{[\chi]}\colon V \to V_{[\chi]}$ onto the corresponding isotypic subspace.*

(3) *If the support $\mathrm{supp}(\pi, V) := \{[\chi] \in \widehat{K}\colon V_{[\chi]} \neq \{0\}\}$ is finite, then $V = \bigoplus_{[\chi] \in \widehat{K}} V_{[\chi]}$ is a finite sum.*

Proof.   The first two assertions follows from the Peter–Weyl Theorem ([HoMo98, Thm. 3.5.1]). To derive (3), we note that $p := \sum_{[\chi] \in \mathrm{supp}(\pi, V)} p_{[\chi]}$ is a continuous projection whose range is $V^{[K]}$, so that (1) implies that p is surjective.   ∎

Lemma 3.2 *For each holomorphic Banach representation (π, V) of the complex torus H, the set $\mathcal{P}(V, \mathfrak{h})$ of $\mathfrak{h}$-weights occurring in V is finite with $V = \bigoplus_{\beta \in \mathfrak{h}^*} V_\beta(\mathfrak{h})$.*

Proof.   The derived representation $\mathrm{L}(\pi)\colon \mathfrak{h} \to \mathfrak{gl}(V)$ is a homomorphism of Banach–Lie algebras. For each character $\chi \in \widehat{T}$ for which $V_\chi(T)$ is non-zero, we therefore have $\|\mathrm{L}(\chi)\| \leq \|\mathrm{L}(\pi)\|$, and since the character group $\widehat{T}$ is a discrete subgroup of the dual $\mathrm{L}(T)^* \cong \mathfrak{h}_{\mathbb{R}}^*$, the set $\mathcal{P}(V, \mathfrak{h})$ is finite. The remaining assertion follows from Theorem 3.1(3).   ∎

Definition 3.3 (The derived representation) Let E be a locally convex space and $V := \mathcal{O}(G, E)$ denote the space of E-valued holomorphic functions on G on which G acts by
$$(\pi(g)f)(h) := f(g^{-1}h).$$
We endow V with the compact open topology, turning it into a locally convex space. Then, for each $f \in V$, the map
$$\pi^f\colon G \to V, \quad g \mapsto \pi(g)f$$

is holomorphic because the map

$$\widehat{\pi}^f \colon G \times G \to E, \quad (g,h) \mapsto (\pi(g)f)(h) = f(g^{-1}h)$$

is holomorphic ([Ne01, Prop. III.13]; [Mue06, Thm. 2.2.5]). We thus obtain an equivariant embedding $V \hookrightarrow \mathcal{O}(G,V), f \mapsto \pi^f$, and from that we derive that

$$\mathrm{Ł}(\pi)(x)f = -X_r f,$$

where $X_r \in \mathcal{V}(G)$ is the right invariant vector field with $X_r(\mathbf{1}) = x$, defines a derived representation of $\mathfrak{g}$ on V. It is called the *derived representation* $\mathrm{Ł}(\pi)$.

Definition 3.4 Let $P \cong L \ltimes U \subseteq G$ be a parabolic subgroup (cf. Proposition 1.12) and (ρ, W) be a holomorphic representation of the weakly root graded Banach–Lie group L on E. We extend this representation in the canonical fashion to a representation ρ of P with $U \subseteq \ker \rho$. We obtain a representation on the space

$$\mathcal{O}_\rho(G,E) := \{f \in \mathcal{O}(G,E) \colon (\forall g \in G)(\forall p \in P) \ f(gp) = \rho(p)^{-1}f(g)\}$$

by $(\pi(g)f)(x) = f(g^{-1}x)$.

Lemma 3.5 *Let (ρ, E) be any holomorphic representation of P. If we endow the space $\mathcal{O}_\rho(G,E)$ with the compact open topology, then the following assertions hold:*

(a) *The action of G_Δ on the complex locally convex space $\mathcal{O}_\rho(G,E)$ is holomorphic.*

(b) *The space $\mathcal{O}_\rho(G,E)^{[\mathfrak{h}]}$ of $\mathfrak{h}$-finite vectors is dense.*

Proof.   (a) is a direct consequence of Theorem B.2 in Appendix B.

(b) First we use [Ne01, Thm. III.11(c)] to see that the locally convex space $\mathcal{O}(G,E)$ is complete because G is Banach and E is complete. In particular, the closed subspace $\mathcal{O}_\rho(G,E)$ is complete.

In view of (a), the complex group $T_\mathbb{C} = H$ acts continuously on the complete locally convex space $\mathcal{O}_\rho(G,E)$, so that the assertion follows from the Peter–Weyl Theorem 3.1 and the fact that T-eigenvectors for a character χ are $\mathfrak{h}$-eigenvectors for $\mathrm{Ł}(\chi) \in \mathfrak{h}^*$. ∎

Definition 3.6 We assign to each $D \in U(\mathfrak{g})$ the right invariant differential operator D_r on G, such that $D \mapsto D_r$ is an anti-homomorphism of associative algebras mapping $x \in \mathfrak{g}$ to X_r, the right invariant vector field with $X_r(\mathbf{1}) = x$. It acts on $\mathcal{O}(G,E)$ by

$$(X_r f)(g) := \frac{d}{dt}\bigg|_{t=0} f(\exp_G(tx)g).$$

Since G is connected, each holomorphic function $f\colon G \to E$ is determined by its jet in the identity. We may represent this jet by a linear map

$$\widetilde{f}\colon U(\mathfrak{g}) \to E, \quad D \mapsto (D_r f)(\mathbf{1}).$$

For $x \in \mathfrak{p}$ we then have

$$\widetilde{f}(Dx) = ((Dx)_r f)(\mathbf{1}) = (X_r(D_r f))(\mathbf{1}) = T_{\mathbf{1}}(D_r f)(x)$$
$$= -\mathrm{L}(\rho)(x)(D_r f)(\mathbf{1}) = -\mathrm{L}(\rho)(x)\widetilde{f}(D),$$

showing that we obtain an injection

$$\Phi\colon \mathcal{O}_\rho(G, E) \hookrightarrow \mathrm{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)_c, \quad \Phi(f)(D) := (D_r f)(\mathbf{1}) \qquad (9)$$

into the coinduced Lie algebra representation. The continuity of $\Phi(f)$ on $U(\mathfrak{g})$ follows from the Taylor expansion of the holomorphic function f in $\mathbf{1}$. The $\mathfrak{g}$-equivariance of this injection follows from

$$\Phi(xf)(D) = -\big((D_r X_r)f\big)(\mathbf{1}) = -\big((xD)_r f\big)(\mathbf{1}) = -\Phi(f)(xD) = (x.\Phi(f))(D).$$

We now come to the main result of this section. Under more restrictive assumptions on (ρ, E), the following theorem will be sharpened in Theorem 4.6 below.

Theorem 3.7 *If G is a connected weakly Δ-graded Lie group, P a parabolic subgroup and (ρ, E) a holomorphic representation of P which is trivial on U, then $\mathcal{O}_\rho(G, E)$ is a finite direct sum of $\mathfrak{h}$-weight spaces.*

If, in addition, P is connected, then $\mathcal{O}_\rho(G, E)$ carries the structure of a holomorphic Banach G-module. If, moreover, G is simply connected,

$$\Phi\colon \mathcal{O}_\rho(G, E) \to \mathrm{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)_c^{[\mathfrak{g}_\Delta]}, \quad \Phi(f)(D) := (D_r f)(\mathbf{1})$$

is an isomorphism of $\mathfrak{g}$-modules.

Proof. In view of Lemma 3.2, E decomposes into finitely many $\mathfrak{h}$-weight spaces, so that Theorem 2.10 implies that the set $\mathcal{P}(\mathcal{O}_\rho(G, E), \mathfrak{h})$ of $\mathfrak{h}$-weights in the submodule $\mathcal{O}_\rho(G, E)^{[\mathfrak{g}_\Delta]}$ of $\mathfrak{g}_\Delta$-finite elements is finite. Note that, in view of Lemma 3.5 and the connectedness of G_Δ, this space coincides with the space of G_Δ-finite elements.

Applying the Peter–Weyl Theorem to a maximal compact subgroup of G_Δ (Lemma 3.5), we see that $\mathcal{O}_\rho(G, E)^{[\mathfrak{g}_\Delta]}$ is dense in $\mathcal{O}_\rho(G, E)$ with respect to the compact open topology. Then we apply the Peter–Weyl Theorem 3.1 to the T-action on $\mathcal{O}_\rho(G, E)$ to obtain that each T-, resp., $\mathfrak{h}$-weight of $\mathcal{O}_\rho(G, E)$ occurs in the dense subspace $\mathcal{O}_\rho(G, E)^{[\mathfrak{g}_\Delta]}$. We conclude that $\mathcal{O}_\rho(G, E)$ is a finite direct sum of $\mathfrak{h}$-weight spaces.

This in turn implies that $\mathcal{O}_\rho(G, E)$ is a locally finite $\mathfrak{g}_\Delta$-module (Lemma 2.7) and hence that $\mathrm{im}(\Phi)$ is contained in the locally $\mathfrak{h}$-finite subspace $V_c := \mathrm{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)_c^{[\mathfrak{h}]}$ of the coinduced representation $\mathrm{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)_c$. Since

$\mathcal{O}_\rho(G, E)$ is locally finite under $\mathfrak{g}_\Delta$, the same holds for its image under Φ, and thus $\text{im}(\Phi) \subseteq V_f := V_c^{[\mathfrak{g}_\Delta]}$.

Now we assume that P is connected. We recall from Theorem 2.12 that V_f is a Banach $\mathfrak{g}$-module. Let $q_G \colon \widetilde{G} \to G$ be the universal covering morphism, $\widetilde{P} := q_G^{-1}(P)_0$ be the identity component of the inverse image of P and $\widetilde{\rho} := \rho \circ q_G|_{\widetilde{P}}$ be the representation of $\widetilde{P}$ on E obtained from ρ. Then the action of $\mathfrak{g}$ on V_f integrates to a unique holomorphic $\widetilde{G}$-representation given by a morphism $\widetilde{G} \to \text{GL}(V_f)$ of Banach–Lie groups ([Bou89, Ch. III, §6.1, Thm 1]. Since $\widetilde{G}$ is connected, the corresponding map $\widetilde{\Phi} \colon \mathcal{O}_{\widetilde{\rho}}(\widetilde{G}, E) \to V_f$ is $\widetilde{G}$-equivariant (cf. [GN09]). From Corollary A.7 in the appendix, we further derive that $V_f \subseteq \text{im}(\widetilde{\Phi})$, which proves surjectivity. Since the pullback map

$$q_G^* \colon \mathcal{O}_\rho(G, E) \to \mathcal{O}_{\widetilde{\rho}}(\widetilde{G}, E), \quad f \mapsto f \circ q_G$$

is an injection whose range coincides with the set of functions constant on the cosets of $\ker q_G$, $\mathcal{O}_\rho(G, E)$ can be identified with the closed subspace of $\ker q_G$-fixed points in the Banach $\widetilde{G}$-module V_f, on which the representation of $\widetilde{G}$ factors through a holomorphic representation of $G \cong \widetilde{G}/\ker q_G$. This completes the proof. ∎

If G is a finite dimensional complex reductive group, then all homogeneous spaces G/P are compact, so that it follows immediately from Liouville's Theorem that all holomorphic functions thereon are constant. If G is infinite dimensional, then so is G/P. In particular, it is never compact (locally compact Banach spaces are finite dimensional). However, it behaves in many respects like a compact flag manifold. Combining the preceding theorem with Remark 2.11, we shall see below that all holomorphic functions on any G/P are constant. This result complements naturally the results of Dineen and Mellon on symmetric Banach manifolds of compact type [DM98]. Such manifolds arise naturally from Banach–Lie groups graded by the root system A_1, and their Jordan theoretic proof essentially reduces matters to the case $G = \text{GL}_2(C(X))$, where X is a compact Hausdorff space. To put the following corollary into proper perspective, one should note that it also applies to all finite dimensional root graded groups which are not reductive. A typical example is $\text{SL}_n(A)$ for any finite dimensional complex algebra A. For $A = \mathbb{C}[\varepsilon]$, $\varepsilon^2 = 0$, we obtain in particular the tangent bundle $T(\mathbb{P}_1(\mathbb{C}))$ as $\text{SL}_2(A)/P$ where $P \subseteq \text{SL}_2(A)$ is the parabolic subgroup of upper triangular matrices (cf. [NS09]).

Corollary 3.8 *If G is a connected root graded Banach–Lie group and $P \subseteq G$ a parabolic subgroup, then all holomorphic functions on G/P are constant.*

Proof. If $q_G \colon \widetilde{G} \to G$ is the universal covering map and $\widetilde{P} := q_G^{-1}(P)$, then $\widetilde{P}$ is a parabolic subgroup of $\widetilde{G}$ with $\widetilde{G}/\widetilde{P} \cong G/P$. We may therefore assume that G is simply connected. If P is not connected, then the canonical map $q \colon G/P_0 \to G/P$ is a covering, and since each holomorphic function on G/P

pulls back to a holomorphic function on G/P_0, we may also assume that P is connected. For the trivial representation $\rho\colon P \to \mathbb{C}^\times \cong \mathrm{GL}(\mathbb{C})$ we then have

$$\mathcal{O}(G/P) \cong \mathcal{O}_\rho(G,\mathbb{C}) \cong \mathrm{Hom}_\mathfrak{p}(U(\mathfrak{g}),\mathbb{C})_c^{[\mathfrak{g}_\Delta]}$$

(Theorem 3.7), and Remark 2.11 implies that this space is one dimensional with trivial G-action. Therefore each holomorphic function on G/P is constant. ∎

Corollary 3.9 *If G is finite dimensional connected, and (ρ, E) is a finite dimensional holomorphic representation of P, then $\dim \mathcal{O}_\rho(G,E) < \infty$.*

Proof. Since $\mathcal{O}_\rho(G,E)$ embeds into $\mathrm{Hom}_\mathfrak{p}(U(\mathfrak{g}),E)_c^{[\mathfrak{g}_\Delta]}$, it suffices to show that the latter space is finite dimensional. In the proof of Theorem 2.12, we obtained the Banach structure on this space by embedding it into $\mathrm{Hom}(U(\mathfrak{n}),E)_c^k$ for some $k \in \mathbb{N}$. If $\mathfrak{g}$, and therefore $\mathfrak{n}$, is finite dimensional, then $\mathrm{Hom}(U(\mathfrak{n}),E)_c^k$ is finite dimensional for each $k \in \mathbb{N}$, and $\mathrm{Hom}_\mathfrak{p}(U(\mathfrak{g}),E)_c^{[\mathfrak{g}_\Delta]}$ inherits this property. ∎

4 Non-connected parabolic subgroups

In Theorem 3.7, we have seen how to identify the space $\mathcal{O}_\rho(G,E)$ with the $\mathfrak{g}_\Delta$-finite submodule of the corresponding coinduced Lie algebra module, provided G is 1-connected and P is connected. If G is connected but not simply connected, then we may always consider the universal covering map $q_G\colon \widetilde{G} \to G$ and consider the parabolic subgroup $\widetilde{P} := q_G^{-1}(P)$ in $\widetilde{G}$. For $\widetilde{\rho} := \rho \circ q_G|_{\widetilde{P}}$, the pullback map then induces a bijection

$$q_G^*\colon \mathcal{O}_\rho(G,E) \to \mathcal{O}_{\widetilde{\rho}}(\widetilde{G},E), \quad f \mapsto f \circ q_G.$$

In fact, since $\ker q_G$ is contained in $\ker \widetilde{\rho}$, each $f \in \mathcal{O}_{\widetilde{\rho}}(\widetilde{G},E)$ is constant on the cosets of $\ker q_G$, hence factors through a function on G, which clearly is contained in $\mathcal{O}_\rho(G,E)$.

This argument shows that the assumption of G being simply connected is quite harmless. However, the connectedness of P is a more tricky issue, and if G is simply connected, then $\pi_0(P) \cong \pi_1(G/P)$ is non-trivial if G/P is not simply connected (cf. Remark 4.1). This happens for many natural examples (see Example 4.9).

Therefore it is desirable to understand the passage from a parabolic subgroup P to its identity component P_0 and, putting $\rho_0 := \rho|_{P_0}$, to understand the difference between $\mathcal{O}_{\rho_0}(G,E)$ and its subspace $\mathcal{O}_\rho(G,E)$. Of particular interest is the question whether the non-triviality of $\mathcal{O}_{\rho_0}(G,E)$ implies that $\mathcal{O}_\rho(G,E)$ is also non-trivial, resp., for which extension ρ of ρ_0, the space $\mathcal{O}_\rho(G,E)$ is non-trivial.

Remark 4.1 Since G is connected, the long exact homotopy sequence of the P-principal bundle $G \to G/P$ provides an exact sequence

$$\pi_1(P) \to \pi_1(G) \to \pi_1(G/P) \to \pi_0(P) \to \mathbf{1},$$

and if G is 1-connected, this leads to $\pi_1(G/P) \cong \pi_0(P)$.

Remark 4.2 If a holomorphic representation $\rho \colon P_1 \to \mathrm{GL}(E)$ extends to a larger subgroup $P \subseteq N_G(P_1)$ containing P_1, then, for each $p \in P$, the representation

$$(p.\rho)(x) := \rho(p^{-1}xp)$$

is equivalent to ρ. Since the equivalence class of $p.\rho$ only depends on the coset $[p] \in P/P_1$, we obtain an action of P/P_1 on the set of equivalence classes of holomorphic representations of P on E.

Remark 4.3 Let $g \in N_G(P)$ and consider the corresponding right multiplication $\overline{\rho}_g \colon G/P \to G/P, xP \mapsto xgP$.

For a holomorphic representation $\rho \colon P \to \mathrm{GL}(E)$ we define the associated holomorphic homogeneous vector bundle

$$\mathbb{E}_\rho := (G \times E)/P := G \times_\rho E$$

whose elements we write as $[g, v]$.

(a) For two holomorphic representations $\rho_1, \rho_2 \colon P \to \mathrm{GL}(E)$, any G-equivariant bundle isomorphism $\varphi \colon \mathbb{E}_{\rho_1} \to \mathbb{E}_{\rho_2}$ covering $\overline{\rho}_g$ is of the form $\varphi([y, v]) = [yg, \psi(v)]$ for some $\psi \in \mathrm{GL}(E)$. Since φ is well-defined, we have for each $p \in P$:

$$[yg, \psi(v)] = \varphi([y, v]) = \varphi([yp, \rho_1(p)^{-1}v]) = [ypg, \psi\rho_1(p)^{-1}v]$$
$$= [yg, \rho_2(g^{-1}pg)\psi\rho_1(p)^{-1}v],$$

which means that

$$\psi \circ \rho_1(p) = \rho_2(g^{-1}pg) \circ \psi \quad \text{for} \quad p \in P, \tag{10}$$

i.e., that ψ intertwines the representations ρ_1 and $g.\rho_2$. If, conversely, ψ satisfies this relation, then $\varphi([y, v]) := [yg, \psi(v)]$ is a well-defined G-equivariant bundle morphism covering $\overline{\rho}_g$.

(b) For $g \in N_G(P)$ and $\psi \in \mathrm{GL}(E)$ we consider the operator

$$M_g \colon \mathcal{O}(G, E) \to \mathcal{O}(G, E), \quad M_g(f) := \psi \circ f \circ \rho_g, \quad \rho_g(x) = xg.$$

If M_g maps $\mathcal{O}_{\rho_1}(G, E)$ into $\mathcal{O}_{\rho_2}(G, E)$, then

$$\rho_2(p)^{-1}\psi f(yg) = \psi f(ypg) = \psi f(ygg^{-1}pg) = \psi\rho_1(g^{-1}pg)^{-1}f(yg)$$

holds for each $f \in \mathcal{O}_{\rho_1}(G, E)$, $y \in G$ and $p \in P$. If this is the case and $\mathrm{ev}_\mathbf{1}|_{\mathcal{O}_{\rho_1}(G,E)}$ has dense range in E, then we obtain $\rho_2(p)^{-1}\psi = \psi\rho_1(g^{-1}pg)^{-1}$, which is equivalent to

$$\rho_1(g^{-1}pg) = \psi^{-1}\rho_2(p)\psi \quad \text{for} \quad p \in P.$$

If, conversely, this condition is satisfied, then the preceding calculation shows that M_g maps $\mathcal{O}_{\rho_1}(G, E)$ into $\mathcal{O}_{\rho_2}(G, E)$.

Proposition 4.4 *Suppose that* $\mathrm{ev}_{\mathbf{1}}\colon \mathcal{O}_\rho(G,E) \to E$ *has dense range and let* $p \in N_G(P)$. *Then the following are equivalent:*

(a) *The two representations ρ and $p.\rho$ define G-equivalent holomorphic vector bundles $\mathbb{E}_\rho$ and $\mathbb{E}_{p.\rho}$.*

(b) *The representations ρ and $p.\rho$ are equivalent.*

(c) *The right multiplication map $\overline{\rho}_p\colon G/P \to G/P, xP \mapsto xgP$ lifts to a G-equivariant bundle isomorphism $M_p\colon \mathbb{E}_\rho \to \mathbb{E}_\rho$.*

(d) *There exists an $A_p \in \mathrm{GL}(E)$ for which the operator*

$$M_p(f) := A_p \circ f \circ \rho_p, \quad \rho_p(x) = xp, \tag{11}$$

preserves $\mathcal{O}_\rho(G,E)$.

Proof. (a) $\Longleftrightarrow$ (b) follows from Remark 4.3(a), applied with $g = \mathbf{1}$, $\rho_1 = \rho$ and $\rho_2 = p.\rho$.
    (b) $\Longleftrightarrow$ (c) follows from Remark 4.3(a), applied to $\rho_1 = \rho_2 = \rho$.
    (b) $\Longleftrightarrow$ (d) follows from Remark 4.3(d).   ■

Corollary 4.5 *If $\chi\colon P \to \mathbb{C}^\times$ is a holomorphic character, $\mathcal{O}_\rho(G) \neq \{0\}$ and $g \in N_G(P)$, then the space $\mathcal{O}_\rho(G)$ is invariant under ρ_g if and only if $g.\rho = \rho$.*

Theorem 4.6 *Assume that G is a connected weakly root graded Banach–Lie group, $P_0 \subseteq G$ is a connected parabolic subgroup, and the holomorphic representation $\rho_0\colon P_0 \to \mathrm{GL}(E)$ is irreducible with $\mathrm{End}_{P_0}(E) = \mathbb{C}\mathbf{1}$ and that $\mathcal{O}_{\rho_0}(G,E) \neq \{0\}$. We further assume that $P \subseteq N_G(P_0) = N_G(\mathfrak{p})$ is an open subgroup satisfying $p.\rho_0 \sim \rho_0$ for each $p \in P$. Then the following assertions hold:*

(a) *There exists a unique extension $\rho\colon P \to \mathrm{GL}(E)$ of ρ_0 satisfying $\mathcal{O}_{\rho_0}(G,E) = \mathcal{O}_\rho(G,E)$. For all other homomorphic extensions $\gamma\colon P \to \mathrm{GL}(E)$, the space $\mathcal{O}_\gamma(G,E)$ is trivial.*

(b) *The map $\Phi\colon \mathcal{O}_\rho(G,E) \to \mathrm{Hom}_{\mathfrak{p}}(U(\mathfrak{g}),E)_c^{[\mathfrak{g}_\Delta]}, \Phi(f)(D) = (D_r f)(\mathbf{1})$ is a linear isomorphism.*

Proof. **Case 1: G is simply connected.** First we use Lemma 3.2 to see that the representation ρ_0 has only finitely many $\mathfrak{h}$-weights. Since G is simply connected, Theorem 3.7 provides a $\mathfrak{g}$-equivariant isomorphism

$$\Phi\colon \mathcal{O}_{\rho_0}(G,E) \to \mathrm{Hom}_{\mathfrak{p}}(U(\mathfrak{g}),E)_c^{[\mathfrak{g}_\Delta]}.$$

Therefore Proposition 2.13(ii) implies that $\mathrm{ev}_{\mathbf{1}}\colon \mathcal{O}_{\rho_0}(G,E)^N \to E$ is a linear isomorphism of the subspace of N-invariant elements onto E. In particular,

$\mathrm{ev}_{\mathbf{1}} \colon \mathcal{O}_{\rho_0}(G, E) \to E$ is surjective. For each $\varphi \in \mathrm{End}_G(\mathcal{O}_{\rho_0}(G, E))$, we now derive from Proposition 2.13(iii),(iv) that the P_0-morphism

$$\mathrm{ev}_{\mathbf{1}} \circ \varphi \colon \mathcal{O}_{\rho_0}(G, E) \to E$$

annihilates $\ker(\mathrm{ev}_{\mathbf{1}})$ and that there exists a $\psi = \alpha\,\mathrm{id}_E \in \mathrm{End}_{\mathfrak{p}}(E) = \mathbb{C}\mathbf{1}$ with

$$\mathrm{ev}_{\mathbf{1}} \circ \varphi = \psi \circ \mathrm{ev}_{\mathbf{1}} = \alpha \cdot \mathrm{ev}_{\mathbf{1}}\,.$$

This proves that, for each $g \in G$, we have

$$\varphi(f)(g) = \mathrm{ev}_{\mathbf{1}}(g^{-1}.\varphi(f)) = \mathrm{ev}_{\mathbf{1}}(\varphi(g^{-1}.f)) = \alpha\,\mathrm{ev}_{\mathbf{1}}(g^{-1}.f) = \alpha f(g),$$

i.e., $\varphi(f) = \alpha f$. We thus obtain

$$\mathrm{End}_G(\mathcal{O}_{\rho_0}(G, E)) = \mathbb{C}\,\mathrm{id}\,. \tag{12}$$

In the following we write $c_g(x) := gxg^{-1}$ for conjugation with g. Now let $p \in P$, pick $A_p \in \mathrm{GL}(E)$ with $p.\rho_0 = c_{A_p} \circ \rho_0$, and define the operator

$$M_p \colon \mathcal{O}_{\rho_0}(G, E) \to \mathcal{O}_{\rho_0}(G, E), \quad M_p f := A_p \circ f \circ \rho_p$$

(Proposition 4.4(d)). Since M_p commutes with the G-action by left translations, there exists an $\alpha_p \in \mathbb{C}$ with $M_p = \alpha_p \mathbf{1}$. This means that each $f \in \mathcal{O}_{\rho_0}(G, E)$ satisfies the equation $f \circ \rho_p = \alpha_p A_p^{-1} \circ f$. Therefore $\rho(p) := \alpha_p^{-1} A_p \in \mathrm{GL}(E)$ satisfies

$$f(xp) = \rho(p)^{-1} f(x) \quad \text{for all} \quad x \in G, f \in \mathcal{O}_{\rho_0}(G, E). \tag{13}$$

Since $\mathrm{ev}_{\mathbf{1}}$ is surjective, evaluating in $x = \mathbf{1}$ shows that this relation determines $\rho(p)$ uniquely. In particular, $\rho(p) = \rho_0(p)$ for $p \in P_0$. We thus obtain a map $\rho \colon P \to \mathrm{GL}(E)$, determined uniquely by (13). For $p_1, p_2 \in P$ we have for each $f \in \mathcal{O}_{\rho_0}(G, E)$

$$\rho(p_1 p_2)^{-1} f(x) = f(x p_1 p_2) = \rho(p_2)^{-1} \rho(p_1)^{-1} f(x),$$

showing that ρ is multiplicative, hence a representation of P on E. Clearly, our construction implies that $\mathcal{O}_\rho(G, E) = \mathcal{O}_{\rho_0}(G, E)$.

If $\gamma \colon P \to \mathrm{GL}(E)$ is another extension of $\rho_0 \colon P_0 \to \mathrm{GL}(E)$, then we also have $\rho_0 \circ c_p = c_{\gamma(p)} \circ \rho_0$ for each $p \in P$, so that the argument in the preceding proof implies that $\gamma = \chi \cdot \rho$ for some map $\chi \colon P \to \mathbb{C}^\times$. Since $\chi(P)$ is central in $\mathrm{GL}(E)$, the fact that γ is a homomorphism implies that χ is a homomorphism, and since γ also extends ρ_0, χ vanishes on P_0. Assume that $\chi(p) \neq 1$ for some $p \in P$.

For any $f \in \mathcal{O}_\gamma(G, E)$, we now have $f \in \mathcal{O}_{\rho_0}(G, E) = \mathcal{O}_\rho(G, E)$, and therefore

$$\chi(p)\rho(p)f(\mathbf{1}) = \gamma(p)f(\mathbf{1}) = f(p^{-1}) = \rho(p)f(\mathbf{1}).$$

Since $\chi(p) \neq 1$, this implies that $f(\mathbf{1}) = 0$, and since $\mathcal{O}_\gamma(G, E)$ is G-invariant, we obtain $f = 0$.

Case 2: G is connected but not simply connected. Let $q_G \colon \widetilde{G} \to G$ be the universal covering, $\widehat{P}_1 := q_G^{-1}(P_0)$ and $\widehat{P} := q_G^{-1}(P)$. We also put $\widehat{\rho}_1 := \rho_0 \circ q_G|_{\widehat{P}_1}$ and $\widehat{\rho}_0 := \widehat{\rho}_1|_{\widehat{P}_0}$. Then we have a linear isomorphism

$$q_G^* \colon \mathcal{O}_{\rho_0}(G, E) \to \mathcal{O}_{\widehat{\rho}_1}(\widetilde{G}, E), \quad f \mapsto f \circ q_G$$

because $\ker q_G \subseteq \widehat{P}_1$ is contained in the kernel of $\widehat{\rho}_1$. From the simply connected case we derive that $\mathcal{O}_{\widehat{\rho}_1}(\widetilde{G}, E) = \mathcal{O}_{\widehat{\rho}_0}(\widetilde{G}, E)$.

Since $\widehat{P}$ satisfies $p.\widehat{\rho}_0 \sim \widehat{\rho}_0$ for each $p \in \widehat{P}$, there exists an extension $\widehat{\rho} \colon \widehat{P} \to \mathrm{GL}(E)$ with

$$\mathcal{O}_{\widehat{\rho}}(\widetilde{G}, E) = \mathcal{O}_{\widehat{\rho}_0}(\widetilde{G}, E).$$

From the uniqueness assertion in Case 1, we derive that $\widehat{\rho}|_{\widehat{P}_1} = \widehat{\rho}_1$, hence that $\ker q_G \subseteq \ker \widehat{\rho}$. Therefore $\widehat{\rho}$ factors through an extension $\rho \colon P \to \mathrm{GL}(E)$ with $\rho \circ q_G = \widehat{\rho}$, for which

$$q_G^* \colon \mathcal{O}_{\rho}(G, E) \to \mathcal{O}_{\widehat{\rho}}(\widetilde{G}, E) = \mathcal{O}_{\widehat{\rho}_0}(\widetilde{G}, E) = q_G^*(\mathcal{O}_{\rho_0}(G, E))$$

is a linear isomorphism. This implies that $\mathcal{O}_{\rho}(G, E) = \mathcal{O}_{\rho_0}(G, E)$ because q_G^* is injective. For any other extension γ of ρ_0 we obtain an extension $\widehat{\gamma} := \gamma \circ q_G$ of $\widehat{\rho}_0$ to $\widehat{P}$ which differs from ρ. Therefore $q_G^*(\mathcal{O}_{\gamma}(G, E)) \subseteq \mathcal{O}_{\widehat{\gamma}}(\widetilde{G}, E) = \{0\}$ implies that $\mathcal{O}_{\gamma}(G, E)$ vanishes.

This completes the proof of (a). To verify (b), we collect the information obtained so far to obtain isomorphisms

$$\mathcal{O}_{\rho}(G, E) \xrightarrow{q_G^*} \mathcal{O}_{\widehat{\rho}}(\widetilde{G}, E) = \mathcal{O}_{\widehat{\rho}_0}(\widetilde{G}, E) \xrightarrow{\Phi_{\widetilde{G}}} \mathrm{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)_c^{[\mathfrak{g}_\Delta]}.$$

Since $\Phi_{\widetilde{G}} \circ q_G^* = \Phi$, this proves (b). ∎

Corollary 4.7 *Assume that G is connected and that the holomorphic character $\chi_0 \colon P_0 \to \mathbb{C}^\times$ satisfies $\mathcal{O}_{\chi_0}(G) \neq \{0\}$. We further assume that $P \subseteq N_G(P_0)$ is an open subgroup fixing χ_0. Then there exists a unique extension $\chi \colon P \to \mathbb{C}^\times$ of ρ_0 to a holomorphic character of P such that*

$$\mathcal{O}_{\chi_0}(G) = \mathcal{O}_{\chi}(G).$$

For all other extensions γ of χ_0, we have $\mathcal{O}_{\gamma}(G) = \{0\}$.

Remark 4.8 If G is finite dimensional and complex reductive, i.e., $Z(G)_0 \cong (\mathbb{C}^\times)^r$ is a complex torus, and E is also finite-dimensional, then $\mathcal{O}_{\rho_0}(G, E)$ is finite dimensional by Corollary 3.9. This implies that the holomorphic representation of G on this space is semisimple, so that the relation (12) from the proof of Theorem 4.6 implies that $\mathcal{O}_{\rho_0}(G, E)$ is an irreducible G-representation.

Example 4.9 For $A = C(\mathbb{S}^1, \mathbb{C})$ and $G = \mathrm{SL}_2(A)_0$ (the identity component), the standard parabolic subgroup P is given by

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in A^\times, b \in A \right\} \cong A \rtimes A^\times,$$

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because for each $a \in A^\times$, the relation

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a^{-1} - 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a - 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -a^{-1} \\ 0 & 1 \end{pmatrix}$$

implies that all diagonal matrices in $\mathrm{SL}_2(A)$ are lying in the identity component. We conclude that $\pi_0(P) \cong \pi_0(A^\times) \cong \mathbb{Z}$. As $A^\times$ is abelian and $A_0^\times$ is divisible, we have

$$A^\times \cong A_0^\times \times \mathbb{Z} \quad \text{and} \quad P \cong P_0 \rtimes \mathbb{Z}.$$

In view of $(P_0, P_0) = A \rtimes \{\mathbf{1}\}$, each character of P_0 factors through $A_0^\times$. We also see that each character has many extensions to a character on P. In view of Theorem 4.6, this implies the existence of extensions χ with

$$\{0\} = \mathcal{O}_\chi(G) \neq \mathcal{O}_{\chi_0}(G).$$

Typical examples arise as follows. If $\chi(a) := a(x)$ arises from evaluation in some $x \in \mathbb{S}^1$, then $\mathcal{O}_\chi(G)$ is non-zero, and the corresponding space of holomorphic sections is isomorphic to $\mathbb{C}^2$, the representation on this space being given by

$$G \to \mathrm{SL}_2(\mathbb{C}), \quad g \mapsto g(x)^{-\top}.$$

The winding number

$$w \colon A^\times \to \mathbb{Z}$$

is a character of $A^\times$, vanishing on $A_0^\times$, for which all other extensions of χ_0 to $A^\times$ are of the form

$$\chi_z(a) = \chi(a) z^{w(a)}$$

for some $z \in \mathbb{C}^\times$. In view of Theorem 4.6, we obtain only for $z = 1$ a non-trivial space of holomorphic sections.

5 Realizing irreducible holomorphic representations

In the preceding sections we developed techniques to study the holomorphic representations of G in the Banach spaces $\mathcal{O}_\rho(G, E)$, defined by a holomorphic representation (ρ, E) of P. In this section we briefly discuss the converse, namely to which extent irreducible holomorphic Banach representations (V, π) of the connected group G have realizations in spaces $\mathcal{O}_\rho(G, P)$, where $U \subseteq \ker \rho$.

Let (V, π) be an irreducible holomorphic Banach representation of G, i.e., all closed G-invariant subspaces of V are trivial. Since the representation is holomorphic, each closed $\mathfrak{g}$-invariant subspace is G-invariant, so that the derived $\mathfrak{g}$-representation on V is also irreducible. We consider a connected parabolic subgroup P of G.

We pick x_Σ as in Remark 1.5 and recall from Lemma 3.2 that V is the direct sum of finitely many $\mathfrak{h}$-weight spaces. Let $\lambda \in \mathbb{R}$ denote the minimal eigenvalue of x_Σ on V. Since $\Sigma^-(x_\Sigma) \leq -1$, we clearly have

$$V_\lambda(x_\Sigma) \subseteq V^{\mathfrak{n}}, \tag{14}$$

and likewise

$$V_{>\lambda}(x_\Sigma) := \sum_{\mu > \lambda} V_\mu(x_\Sigma) \supseteq \mathfrak{u}.V. \tag{15}$$

Since x_Σ is central in $\mathfrak{l}$, the closed subspace $V_\lambda(x_\Sigma)$ is $\mathfrak{l}$-invariant. For each closed $\mathfrak{l}$-invariant subspace $W \subseteq V_\lambda(x_\Sigma) \subseteq V^{\mathfrak{n}}$, the PBW Theorem implies that

$$U(\mathfrak{g})W = U(\mathfrak{u})U(\mathfrak{l})U(\mathfrak{n})W = U(\mathfrak{u})W \subseteq W + \mathfrak{u}.V \subseteq W \oplus V_{>\lambda}(x_\Sigma).$$

Therefore the irreducibility of the $\mathfrak{g}$-module V implies that $V_\lambda(x_\Sigma)$ is an irreducible $\mathfrak{l}$-module. Further, the density of $U(\mathfrak{g})V_\lambda(x_\Sigma)$ in V implies the equality

$$\overline{\mathfrak{u}.V} = V_{>\lambda}(x_\Sigma). \tag{16}$$

Since $\overline{\mathfrak{u}.V}$ is $\mathfrak{p}$-invariant, the quotient space $E := V/\overline{\mathfrak{u}.V}$ inherits a natural $\mathfrak{p}$-module structure, where $\mathfrak{u}$ acts trivially. We write $\beta \colon V \to E$ for the corresponding quotient map. Let ρ denote the corresponding representation of P on this Banach space. Then Theorem A.6(2) yields an embedding

$$\beta_G \colon V \hookrightarrow \mathcal{O}_\rho(G, E), \quad \beta_G(v)(g) := \beta(g^{-1}.v).$$

Clearly, β_G maps $V_\lambda(x_\Sigma)$ into the space of N-invariants in $\mathcal{O}_\rho(G, E)$ and satisfies $\mathrm{ev}_{\mathbf{1}} \circ \beta_G = \beta$, so that $\mathrm{ev}_1 \colon \mathcal{O}_\rho(G, E) \to E$ is surjective. Therefore

$$\beta_G(V_\lambda(x_\Sigma)) = \mathcal{O}_\rho(G, E)^N \cong E.$$

To see that β_G is continuous with respect to the natural Banach structure on $\mathcal{O}_\rho(G, E)$, we observe that for $x_1, \ldots, x_k \in \mathfrak{g}$, we have

$$\Phi(\beta_G(v))(x_1 \cdots x_k) = ((X_k)_r \cdots (X_1)_r \beta_G(v))(\mathbf{1}) = (-1)^k \beta(x_k \cdots x_1.v),$$

which defined a continuous k-linear map $\mathfrak{g}^k \to E$. From the continuity of β_G we finally derive that its image is contained in the closed G-submodule of $\mathcal{O}_\rho(G, E)$ generated by the subspace $\mathcal{O}_\rho(G, E)^N \cong E$ of N-invariants. (Proposition 2.13(ii)).

Putting all this together, we have:

Theorem 5.1 *Let G be a connected weakly root graded Banach–Lie group and (π, V) be an irreducible holomorphic representation of G. If P is a connected parabolic subgroup of G, then we obtain an irreducible holomorphic representation (ρ, E) on $E := V/\overline{\mathfrak{u}.V}$ with $U \subseteq \ker \rho$. If $\beta \colon V \to E$ denotes the quotient map, then*

$$\beta_G \colon V \to \mathcal{O}_\rho(G, E), \quad \beta_G(v)(g) := \beta(g^{-1}.v)$$

defines a continuous morphism of holomorphic Banach G-modules whose image is a dense subspace in the minimal closed G-submodule of $\mathcal{O}_\rho(G, E)$.

A   Generalities on holomorphic induction

The following theorem is the natural version of Frobenius reciprocity for holo-morphically induced representations. The assumptions on the G-module W and the P-module E are rather weak, so that they do not only apply to Banach representations, but also to representations in spaces of holomorphic functions.

Theorem A.1 (Frobenius Reciprocity) *Let W be a G-representation with holo-morphic orbit maps and (ρ, E) a locally convex representation of P with holo-morphic orbit maps. Then the map*

$$\mathrm{ev_1} \circ \colon \ \mathrm{Hom}_G(W, \mathcal{O}_\rho(G, E)) \to \mathrm{Hom}_P(W, E)$$

is bijective if Hom_P denotes the set of continuous P-morphisms and Hom_G the set of those G-morphisms that are continuous with respect to the topology of pointwise convergence on $\mathcal{O}_\rho(G, E)$. Its inverse is given by

$$\mathrm{Hom}_P(W, E) \to \mathrm{Hom}_G(W, \mathcal{O}_\rho(G, E)), \quad \beta \mapsto \beta_G, \quad \beta_G(w)(g) = \beta(g^{-1}.w).$$

Proof.   Note that

$$\mathrm{ev_1} \colon \mathcal{O}_\rho(G, E) \to E, \quad f \mapsto f(\mathbf{1})$$

is P-equivariant:

$$(p.f)(\mathbf{1}) = f(p^{-1}) = \rho(p).f(\mathbf{1})$$

and continuous with respect to the topology of pointwise convergence, so that $\mathrm{ev_1} \circ$ maps $\mathrm{Hom}_G(W, \mathcal{O}_\rho(G, E))$ into $\mathrm{Hom}_P(W, E)$.

Let $\varphi \in \mathrm{Hom}_G(W, \mathcal{O}_\rho(G, E))$ and assume that $\mathrm{ev_1} \circ \varphi = 0$, i.e.,

$$\varphi(w)(\mathbf{1}) = 0 \quad \text{for each } w \in W.$$

Since φ is G-equivariant, $\varphi(w)$ vanishes on all of G, so that $\varphi = 0$. Therefore $\mathrm{ev_1} \circ$ is injective.

To see that it is also surjective, let $\beta \in \mathrm{Hom}_P(W, E)$ and define

$$\beta_G \colon W \to \mathcal{O}(G, E), \quad \beta_G(w)(g) := \beta(g^{-1}.w)$$

(recall that G-orbit maps in W are holomorphic). Then

$$\beta_G(w)(gp) = \beta(p^{-1}g^{-1}w) = \rho(p)^{-1}\beta(g^{-1}w)$$

implies that $\beta_G(W) \subseteq \mathcal{O}_\rho(G, E)$, and it is clear that β_G is G-equivariant with $\mathrm{ev_1} \circ \beta_G = \beta$. Finally we note that β_G is continuous with respect to the topology of pointwise convergence on $\mathcal{O}(G, E)$ since β is continuous. ∎

Remark A.2 (a) Note that the orbit maps for the G-action on the space $\mathcal{O}_\rho(G, E)$, endowed with the compact open topology, are holomorphic because for each $f \in \mathcal{O}_\rho(G, E)$, the map

$$G \times G \to E, \quad (x, y) \mapsto f(x^{-1}y)$$

is holomorphic ([Ne01, Prop. III.13]).

(b) In view of (a), each G-submodule W of $\mathcal{O}_\rho(G, E)$ satisfies the assumptions of Theorem A.1, so that all G-morphisms $W \to \mathcal{O}_\rho(G, E)$, continuous w.r.t. the topology of pointwise convergence, correspond to continuous P-morphisms $W \to E$.

Remark A.3 Let (π, V) he a representation of G with holomorphic orbit maps and
$$\mathcal{O}_\pi(G, V) := \{f \in \mathcal{O}(G, V) : (\forall g, x \in G) \ f(xg) = \pi(g)^{-1} f(x)\},$$

then the evaluation map

$$\mathrm{ev}_\mathbf{1} : \mathcal{O}_\pi(G, V) \to V, \quad f \mapsto f(\mathbf{1})$$

is a G-equivariant isomorphism whose inverse is given by $v \mapsto f_v$ with $f_v(g) = \pi(g^{-1})v$. Therefore we may identify $\mathcal{O}_\pi(G, V)$ with V. If we endow $\mathcal{O}_\pi(G, V)$ wit the topology of pointwise or compact convergence, we even obtain a topological isomorphism $V \cong \mathcal{O}_\pi(G, V)$.

Corollary A.4 $\mathcal{O}_\rho(G, E)$ *is non-zero if and only there exists some holomorphic G-module W and a non-zero continuous P-homomorphism $W \to E$.*

Example A.5 Let A be a unital Banach algebra and consider the root graded Banach–Lie group $G := \mathrm{GL}_n(A)$. Then the space

$$\mathfrak{p} = \{(x_{ij}) \in \mathfrak{gl}_n(A) : i > j \Rightarrow x_{ij} = 0\}$$

of upper triangular matrices is a parabolic subalgebra and

$$P := \mathfrak{p} \cap \mathrm{GL}_n(A)$$

is a corresponding parabolic subgroup.

The preceding corollary provides a rich supply of holomorphic representations of (ρ, E) of P for which $\mathcal{O}_\rho(G, E)$ is non-trivial. In the holomorphic G-module $W = A^n$, the subspaces

$$F_i := \sum_{j \leq i} A e_j, \quad e_j = (\delta_{ij})_{i=1,\ldots,n},$$

are P-invariant, so that each space $E_i := A^n / F_i$ carries a natural holomorphic P-representation ρ_i for which $\mathcal{O}_{\rho_i}(G, E_i) \neq \{0\}$. Identifying E_i in the natural fashion with A^{n-i}, the representation ρ_i is given by $\rho(g_{kl}) = (g_{kl})_{i+1 \leq k, l \leq n}$, resp.,

$$\rho \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = d,$$

if we write $X \in M_n(A)$ as a block matrix with entries in $M_{i,i}(A)$, $M_{i,n-i}(A)$, $M_{n-i,i}(A)$ and $M_{n-i,n-i}(A)$, respectively.

Theorem A.6 *Let $P \subseteq G$ be a connected complex Lie subgroup, (π, V) a G-representation with holomorphic orbit maps and (ρ, E) a P-representation with holomorphic orbit maps. Then the following are equivalent:*

(1) *There exists a G-cyclic continuous $\beta \in \operatorname{Hom}_P(V, E)$, i.e., $\beta(G.v) = \{0\}$ implies $v = 0$.*

(2) *There is a G-equivariant injection $\beta_G \colon V \hookrightarrow \mathcal{O}_\rho(G, E)$ which is continuous with respect to the pointwise topology on $\mathcal{O}_\rho(G, E)$.*

(3) *V embeds into $\operatorname{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)$ such that $\operatorname{ev}_{\mathbf{1}} \colon V \to E$ is continuous.*

Proof. The equivalence of (1) and (2) follows from Theorem A.1.

(2) $\Rightarrow$ (3): For this implication, we only have to recall the inclusion

$$\Phi \colon \mathcal{O}_\rho(G, W) \hookrightarrow \operatorname{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E), \quad \Phi(f)(D) := (D_r f)(\mathbf{1})$$

from Definition 3.6.

(3) $\Rightarrow$ (1): The continuous linear map $\beta := \operatorname{ev}_{\mathbf{1}} |_V \colon V \to W$ is $\mathfrak{p}$-equivariant because

$$\operatorname{ev}_{\mathbf{1}}(p.\varphi) = -\varphi(p) = \mathrm{L}(\rho)(p)\varphi(\mathbf{1}) = \mathrm{L}(\rho)(p)\operatorname{ev}_{\mathbf{1}}(\varphi),$$

hence P-equivariant because P is connected (cf. [GN09]). Further, β is $\mathfrak{g}$-cyclic because $0 = \beta(U(\mathfrak{g}).f) = f(U(\mathfrak{g}))$ implies $f = 0$. Since the G-orbit maps in V are holomorphic, β is also G-cyclic. ∎

Corollary A.7 *Suppose that P is connected and that (ρ, E) is a P-representation with holomorphic orbit maps. Then the image of the Taylor series map*

$$\Phi \colon \mathcal{O}_\rho(G, E) \to \operatorname{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E), \quad \Phi(f)(D) := (D_r f)(\mathbf{1})$$

is the largest $\mathfrak{g}$-submodule V of $\operatorname{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)$ on which the $\mathfrak{g}$-module structure integrates to a G-representation such that, for each $v \in V$, the map

$$G \to E, \quad g \mapsto (g.v)(\mathbf{1})$$

is holomorphic.

Proof. We recall from Definition 3.6 that Φ is injective and $\mathfrak{g}$-equivariant. Let $V \subseteq \operatorname{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)$ be a $\mathfrak{g}$-submodule on which the representation integrates to a G-representation with the required properties. For each $v \in V$, we then obtain a holomorphic function $f_v \in \mathcal{O}(G, E)$ by $f_v(g) := (g^{-1}.v)(\mathbf{1})$. Then $X_r f_v = -f_{x.v}$ holds for each $x \in \mathfrak{g}$, and therefore

$$D_r f_v = f_{D^\sigma.v},$$

where $\sigma \colon U(\mathfrak{g}) \to U(\mathfrak{g}), D \mapsto D^\sigma$, is the unique antiautomorphism with $x^\sigma = -x$ for $x \in \mathfrak{g}$. We now obtain

$$T(f_v)(D) = (D_r f_v)(\mathbf{1}) = f_{D^\sigma.v}(\mathbf{1}) = (D^\sigma.v)(\mathbf{1}) = v(D),$$

so that $T(f_v) = v$. If $v \in V$, then $G.v \subseteq \mathrm{Hom}_{\mathfrak{p}}(U(\mathfrak{g}), E)$, so that we have for each $x \in \mathfrak{p}$

$$(X_l f_v)(g) = (-x.(g^{-1}.v))(\mathbf{1}) = (g^{-1}.v)(x) = -x.\big((g^{-1}.v)(\mathbf{1})\big) = -x.f_v(g),$$

and since P is connected, we obtain $f_v \in \mathcal{O}_\rho(G, E)$ (cf. [GN09]). ∎

B   A general continuity lemma

Lemma B.1 *Let M be a Hausdorff space, V a locally convex space, and S a locally compact topological semigroup which acts continuously on M from the right. Then the induced action*

$$\Phi \colon S \times C(M, V)_c \to C(M, V)_c, \quad (s, f) \mapsto (x \mapsto f(x.s))$$

is continuous with respect to the compact-open topology on the function space.

Proof.   ([Mue06, Lemma 2.2.6]) Consider $s \in S$ and $f \in C(M, V)_c$. The basic open neighborhoods of the image $\varphi := \Phi(s, f)$ are of the form

$$U_{p, K, \varepsilon}(\varphi) = \{g \in C(M, V) \mid p(g(x) - \varphi(x)) < \varepsilon \text{ for all } x \in K\},$$

where p is a continuous seminorm on V, $K \subseteq M$ is a compact subset, and $\varepsilon > 0$. We need to find an open neighborhood of (s, f) in $S \times C(M, V)$ that is mapped completely into $U_{p, K, \varepsilon}(\varphi)$ by Φ. To do this, we look at the following inequality for $t \in S$ and $g \in C(M, V)_c$:

$$p((t.g)(x) - \varphi(x)) \le p(g(x.t) - f(x.t)) + p(f(x.t) - f(x.s))$$

By the continuity of the action of S on M, the continuity of f, and the continuity of p, there is an open neighborhood W of s in the locally compact semigroup S with compact closure $C = \overline{W}$ such that the second term

$$p(f(x.t) - f(x.s))$$

is strictly less than $\varepsilon/2$ for all $t \in W$ and $x \in K$.

Next, we note that the set $K.C$ is a compact subset of M, being the image of $K \times C$ under the continuous action map $M \times S \to M$. For all $g \in U_{p, K.C, \varepsilon/2}(f)$, we then have $p(g(x.t) - f(x.t)) < \varepsilon/2$ for all $t \in C$ and $x \in K$, by definition. We conclude that $p((t.g)(x) - \varphi(x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for all $t \in W$ and $g \in U_{p, K.C, \varepsilon/2}(f)$, which shows

$$\Phi\big(W \times U_{p, K.C, \varepsilon/2}(f)\big) \subseteq U_{p, K, \varepsilon}(\Phi(f, s)),$$

that is, continuity of Φ. ∎

Theorem B.2 *Let M be a complex Banach manifold, V be a complete locally convex space, G be a* finite dimensional *complex Lie group, and $G \times M \to M$ a holomorphic right action. Then the action*

$$G \times \mathcal{O}(M, V) \to \mathcal{O}(M, V), \quad (g.f)(x) = f(g^{-1}.x)$$

is holomorphic with respect to the compact open topology on $\mathcal{O}(M, V)$.

Proof. ([Mue06, Lemma 2.2.6]) This is a less general version of [Ne01, Theorem III.14], so the proof can be found there. However, the proof uses the defective Lemma III.2(iii) from the same source, which we can now replace with our Lemma B.1. ∎

References

[ABG00]   Allison, B. N., Benkart, G., and Y. Gao, *Central extensions of root graded Lie algebras*, Math. Ann. **316** (2000), 499–527

[ABG02]   —, "Lie Algebras graded by the Root System BC_r, $r \geq 2$," Memoirs of the Amer. Math. Soc. **751**, Providence R.I., 2002

[BZ96]   Benkart, G., and E. Zelmanov, *Lie algebras graded by finite root systems and intersection matrix algebras*, Invent. Math. **126** (1996), 1–45

[BM92]   Berman, S., and R. V. Moody, *Lie algebras graded by finite root systems and the intersection matrix algebras of Slodowy*, Invent. Math. **108** (1992), 323–347

[BN04]   Bertram, W., K.-H. Neeb, *Projective completions of Jordan pairs, Part I. The generalized projective geometry of a Lie algebra*, J. Algebra **277:2** (2004), 474–519

[Bou89]   Bourbaki, N., "Lie Groups and Lie Algebras, Chapters 1-3," Springer-Verlag, 1989

[DM98]   Dineen, S., and P. Mellon, *Holomorphic functions on symmetric Banach manifolds of compact type are constant*, Math. Z. **229** (1998), 753–765

[EK64]   van Est, W. T., Korthagen, Th. J., *Non-enlargible Lie algebras*, Ned. Acad. v. Wet., Proc. **A 67**=Indag. Math. **26** (1964), 15–31

[GN09]   Glöckner, H., and K.-H. Neeb, "Infinite dimensional Lie groups, Vol. I, Basic Theory and Main Examples," book in preparation

[HoMo98]   Hofmann, K. H., and S. A. Morris, "The Structure of Compact Groups," Studies in Math., de Gruyter, Berlin, 1998

[Ka90]   Kac, V. G., "Infinite Dimensional Lie Algebras," Cambridge University Press, 1990

[KM97]   Kriegl, A., and P. Michor, "The Convenient Setting of Global Analysis," Math. Surveys and Monographs **53**, Amer. Math. Soc., 1997

[Ms62]   Maissen, B., *Lie-Gruppen mit Banachräumen als Parameterräume*, Acta Math. **108** (1962), 229–270

[MP95]   Moody, R., and A. Pianzola, "Lie algebras with triangular decompositions", Canad. Math. Soc. Series of Monographs and advanced texts, Wiley Interscience, 1995

[Mue06]   Müller, Chr., "Representations of root graded Banach–Lie groups", Thesis, TU Darmstadt, 2006

[Ne00]   Neeb, K.-H., "Holomorphy and Convexity in Lie Theory," Expositions in Mathematics **28**, de Gruyter Verlag, Berlin, 2000

[Ne01]   —, *Representations of infinite dimensional groups*, in "Infinite Dimensional Kähler Manifolds," A. Huckleberry, T. Wurzbacher (Eds.), DMV-Seminar **31**, Birkhäuser Verlag, 2001; 131–178

[Ne03]   —, *Locally convex root graded Lie algebras*, Travaux mathématiques **14** (2003), 25–120

[Ne04]   —, "Infinite dimensional Lie groups and their representations", in "Lie Theory: Lie Algebras and Representations", Progress in Math. **228**, Ed. J. P. Anker, B. Ørsted, Birkhäuser Verlag, 2004; 213–328

[Ne06]   —, *Towards a Lie theory of locally convex groups*Jap. J. Math. 3rd ser. **1:2** (2006), 291–468

[NS09]   Neeb, K.-H., and H. Seppänen, *Borel-Weil Theory for finite dimensional root graded Lie groups*, in preparation